On Constructibility Results for a Class of Non-Selfadjoint Analytic Perturbations of Matrices with Degenerate Eigenvalues

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A₀ has a **Degenerate Eigenvalue** λ₀ if $\det(λ₀ I - A₀) = 0$ and $\frac{d \det(λI - A₀)}{dλ} |_{λ=λ₀} = 0$. 
Definitions
Degenerate Eigenvalue, Analytic Perturbation, Constructibility

- $A_0$ has a **Degenerate Eigenvalue** $\lambda_0$ if $\det(\lambda_0 I - A_0) = 0$ and $\frac{d \det(\lambda I - A_0)}{d\lambda} |_{\lambda = \lambda_0} = 0$.

- $A(\varepsilon)$ is an **Analytic Perturbation** of $A_0$ if $A(0) = A_0$ and $A(\varepsilon) = \sum_{k=0}^{\infty} A_k \varepsilon^k$, $|\varepsilon| << 1$, where $\{A_k\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n \times n}$.
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- **Constructibility**: Compute locally eigenvalues/eigenvectors of $A(\varepsilon)$, 
  for those eigenvalues near $\lambda_0$, with explicit recursive formulas using 
  \( \{A_k\}_{k=0}^{\infty} \).
Definitions
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Constructibility: Compute locally eigenvalues/eigenvectors of $A(\varepsilon)$, for those eigenvalues near $\lambda_0$, with explicit recursive formulas using $\{A_k\}_{k=0}^{\infty}$. 
A Puiseux series is a multivalued convergent series of the form

\[ \alpha(\varepsilon) = \sum_{k=0}^{\infty} \alpha_k \varepsilon^{k/m}, \]

where \( \{\alpha_k\}_{k=0}^{\infty} \), its series coefficients, is a sequence in a normed space.
Definition (Puiseux series)

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The series

\[ \alpha_h(\varepsilon) = \sum_{k=0}^{\infty} \alpha_k (\zeta^h \varepsilon^{1/m})^k, \quad h = 0, \ldots, m - 1, \]

are called the branches of the Puiseux series, where \( \zeta = e^{\frac{2\pi i}{m}} \) and \( i = \sqrt{-1} \).
Definition (Eigenvalue/Eigenvector Puiseux series)

Let $A(\varepsilon)$ be an analytic perturbation of a matrix $A_0$. We say that

$$(\lambda(\varepsilon), x(\varepsilon)) := \left( \sum_{k=0}^{\infty} \lambda_k \varepsilon^{k/m}, \sum_{k=0}^{\infty} x_k \varepsilon^{k/m} \right)$$

is an eigenpair Puiseux series of $A(\varepsilon)$ with period $m$ and $(\lambda_h(\varepsilon), x_h(\varepsilon)), h = 0, \ldots, m - 1$, are its branches provided

$$A(\varepsilon)x_h(\varepsilon) = \lambda_h(\varepsilon)x_h(\varepsilon), h = 0, \ldots, m - 1,$$

for $|\varepsilon| << 1$. We then say that $\lambda(\varepsilon)$ is eigenvalue Puiseux series for $A(\varepsilon)$ with $x(\varepsilon)$ a corresponding eigenvector Puiseux series.
Definition (Complete Collection of Eigenpair Puiseux series)

Let \( A(\varepsilon) \) be an analytic perturbation of a matrix \( A_0 \). A finite collection \( \mathcal{C} \) of eigenpair Puiseux series of \( A(\varepsilon) \) is called complete (with respect to its geometric eigenspaces) if for \( 0 < ||\varepsilon|| << 1 \) and for any eigenpair \( (\lambda, x) \) of \( A(\varepsilon) \) there exists a finite number of eigenpair Puiseux series \( (\lambda(\varepsilon), x^{[1]}(\varepsilon)), \ldots, (\lambda(\varepsilon), x^{[g]}(\varepsilon)) \in \mathcal{C} \) such that for some branch \( \lambda_h(\varepsilon) \) of \( \lambda(\varepsilon) \) we have

1. \( \lambda_h(\varepsilon) = \lambda \),
2. \( x \in \text{span} \left\{ x^{[1]}_h(\varepsilon), \ldots, x^{[g]}_h(\varepsilon) \right\} = \ker(\lambda_h(\varepsilon)I - A(\varepsilon)) \),
3. the vectors \( x^{[1]}_h(\varepsilon), \ldots, x^{[g]}_h(\varepsilon) \) are linearly independent.
Let $A(\varepsilon)$ be an analytic perturbation of a matrix $A_0$. Then there exists a finite collection $\mathcal{C}$ of eigenpair Puiseux series of $A(\varepsilon)$ that is complete.
Theorem ((['85] H. Baumgärtel) Existence of a Complete Collection)

Let $A(\varepsilon)$ be an analytic perturbation of a matrix $A_0$. Then there exists a finite collection $\mathcal{C}$ of eigenpair Puiseux series of $A(\varepsilon)$ that is complete. Moreover, if $\lambda_0$ is an eigenvalue of $A_0$ and we define $\mathcal{C}(\lambda_0)$, the $\lambda_0$-eigenpair group, to be the collection of all those eigenpairs from $\mathcal{C}$ whose eigenvalue Puiseux series have center $\lambda_0$ then $\mathcal{C}(\lambda_0) \neq \emptyset$. 
Let $A(\varepsilon)$ be an analytic perturbation of a matrix $A_0$. Then there exists a finite collection $\mathcal{C}$ of eigenpair Puiseux series of $A(\varepsilon)$ that is complete. Moreover, if $\lambda_0$ is an eigenvalue of $A_0$ and we define $\mathcal{C}(\lambda_0)$, the $\lambda_0$-eigenpair group, to be the collection of all those eigenpairs from $\mathcal{C}$ whose eigenvalue Puiseux series have center $\lambda_0$ then $\mathcal{C}(\lambda_0) \neq \emptyset$. Furthermore, the number of eigenpairs and the set of periods in each $\mathcal{C}(\lambda_0)$ is unique (independent of the choice of $\mathcal{C}$).
Let $A(\varepsilon)$ be an analytic perturbation of a matrix $A_0$ with a degenerate eigenvalue $\lambda_0$. From just the derivatives of $A(\varepsilon)$ at $\varepsilon = 0$, can we determine the number of eigenpairs in $\mathcal{C}(\lambda_0)$ along with their periods and choose $\mathcal{C}(\lambda_0)$ in such a way that there exists explicit recursive formulas to compute all the series coefficients of the eigenpairs in $\mathcal{C}(\lambda_0)$?
Solution $\Rightarrow$ Constructibility
Main Problem
Solution ⇒ Constructibility

- Solution ⇒ Constructibility
- Solution Exist?
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- Solution Exist? → YES [(‘09) A. Welters], for a Class of Perturbations satisfying the Generic Condition
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- Example Application
Solution $\Rightarrow$ Constructibility

Solution Exist? $\rightarrow$ YES [(‘09) A. Welters], for a Class of Perturbations satisfying the Generic Condition

Example Application $\rightarrow$ Study of Slow Light in Photonic Crystals [(‘06) A. Figotin and I. Vitebskiy]
Study of Slow Light in Photonic Crystals $\rightarrow$ a Class of Analytic Perturbations
Study of Slow Light in Photonic Crystals $\rightarrow$ a Class of Analytic Perturbations $\leftrightarrow$ Generic Condition.
Study of Slow Light in Photonic Crystals → a Class of Analytic Perturbations ↔ Generic Condition.

**Definition (Generic Condition)**

Let $A(\varepsilon)$ be an analytic perturbation of a matrix $A_0$ eigenvalue $\lambda_0$. If the characteristic polynomial of $A(\varepsilon)$ evaluated at $\lambda_0$ has a simple zero at $\varepsilon = 0$, that is,

$$\left. \frac{\partial \det(\lambda I - A(\varepsilon))}{\partial \varepsilon} \right|_{(\varepsilon, \lambda) = (0, \lambda_0)} \neq 0,$$

then we say that the analytic perturbation $A(\varepsilon)$ satisfies the **generic condition** at $\lambda_0$. 
Definition (Jordan Block)

We define a $m \times m$ Jordan block corresponding to the eigenvalue $\lambda_0$ to be the $m \times m$ matrix

$$J_m(\lambda_0) := \begin{bmatrix}
\lambda_0 & 1 \\
\lambda_0 & 1 \\
\vdots & \ddots & \ddots \\
\lambda_0 & & & & 1
\end{bmatrix}$$

where the blank entries are zero.

- $J_1(\lambda_0) = [\lambda_0]$, $J_2(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$, $J_3(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix}$, \ldots
Theorem (('09) A. Welters)

Let \( A(\varepsilon) \) be an analytic perturbation of an \( n \times n \) matrix \( A_0 \) with an eigenvalue \( \lambda_0 \) of algebraic multiplicity \( m \). If \( A(\varepsilon) \) satisfies the generic condition,

\[
\frac{\partial \det(\lambda I - A(\varepsilon))}{\partial \varepsilon} \bigg|_{(\varepsilon, \lambda)=(0, \lambda_0)} \neq 0,
\]

then there exists an \( n \times n \) invertible matrix \( U \) and a \((n - m) \times (n - m)\) matrix \( W_0 \), such that

\[
U^{-1}A_0U = \begin{bmatrix} J_m(\lambda_0) & \vdots \\ \vdots & W_0 \end{bmatrix}
\]

and \( \lambda_0 \) is not an eigenvalue of \( W_0 \). Generically, the converse is true.
We denote the standard inner product by \( \langle x, y \rangle := x^* y, \quad x, y \in \mathbb{C}^{n \times 1} \).

Denote the first \( m \) columns of \( U \), \( (U^{-1})^* \) by \( u_1, \ldots, u_m, v_1, \ldots, v_m \), respectively.

Define a matrix \( \Lambda \in \mathbb{C}^{n \times n} \) by

\[
\Lambda := U \left[ \begin{array}{c|c}
J_m(0)^* & \left( W_0 - \lambda_0 I_{n-m} \right)^{-1}
\end{array} \right] U^{-1}.
\]
Main Results
Coefficients up to 2nd Order

Theorem ((('09 A. Welters) Puiseux Series Coefficients up to 2nd Order)

Let \( A(\varepsilon) \) be an analytic perturbation of an \( n \times n \) matrix \( A_0 \) with a degenerate eigenvalue \( \lambda_0 \) of algebraic multiplicity \( m \geq 2 \). Suppose that \( A(\varepsilon) \) satisfies the generic condition at \( \lambda_0 \). Then

\[
\mathcal{C}(\lambda_0) = \left\{ \left( \sum_{k=0}^{\infty} \lambda_k \varepsilon^{k/m}, \sum_{k=0}^{\infty} x_k \varepsilon^{k/m} \right) \right\};
\]
### Theorem ((('09 A. Welters) Puiseux Series Coefficients up to 2nd Order)

Let $A(\varepsilon)$ be an analytic perturbation of an $n \times n$ matrix $A_0$ with a degenerate eigenvalue $\lambda_0$ of algebraic multiplicity $m \geq 2$. Suppose that $A(\varepsilon)$ satisfies the generic condition at $\lambda_0$. Then

1. $\mathcal{C}(\lambda_0) = \left\{ \left( \sum_{k=0}^{\infty} \lambda_k \varepsilon^{k/m}, \sum_{k=0}^{\infty} x_k \varepsilon^{k/m} \right) \right\}$;

2. $\lambda_1^m = \langle v_m, \frac{dA}{d\varepsilon} (0) u_1 \rangle = -\frac{\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}}{\frac{\partial^m}{\partial \lambda^m} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda)=(0, \lambda_0)}} \neq 0$;
Theorem ((('09) A. Welters) Puiseux Series Coefficients up to 2nd Order)

Let $A(\varepsilon)$ be an analytic perturbation of an $n \times n$ matrix $A_0$ with a degenerate eigenvalue $\lambda_0$ of algebraic multiplicity $m \geq 2$. Suppose that $A(\varepsilon)$ satisfies the generic condition at $\lambda_0$. Then

1. $C(\lambda_0) = \left\{ \left( \sum_{k=0}^{\infty} \lambda_k \varepsilon^{k/m}, \sum_{k=0}^{\infty} x_k \varepsilon^{k/m} \right) \right\}$;

2. $\lambda_1^m = \langle v_m, \frac{dA}{d\varepsilon} (0) u_1 \rangle = -\left( \frac{\partial}{\partial \varepsilon} \left. \det(\lambda I - A(\varepsilon)) \right|_{(\varepsilon, \lambda) = (0, \lambda_0)} \right) - \left( \frac{\partial^m}{\partial \lambda^m} \left. \det(\lambda I - A(\varepsilon)) \right|_{(\varepsilon, \lambda) = (0, \lambda_0)} \right) \neq 0$;

3. We may choose $\lambda_1 := \langle v_m, \frac{dA}{d\varepsilon} (0) u_1 \rangle^{1/m}$, for any fixed $m$th root, and $\{x_k\}_{k=0}^{\infty}$ to satisfy the normalization conditions $\langle v_1, x_0 \rangle = 1$, $\langle v_1, x_k \rangle = 0$, for $k \geq 1$. In which case, $\{\lambda_k\}_{k=0}^{\infty}$, $\{x_k\}_{k=0}^{\infty}$ are uniquely determined by explicit recursive formulas in the derivatives of $A(\varepsilon)$ at $\varepsilon = 0$ and ...
Main Results
Coefficients up to 2nd Order

Theorem ((('09) A. Welters) Puiseux Series Coefficients up to 2nd Order)

Let $A(\varepsilon)$ be an analytic perturbation of an $n \times n$ matrix $A_0$ with a degenerate eigenvalue $\lambda_0$ of algebraic multiplicity $m \geq 2$. Suppose that $A(\varepsilon)$ satisfies the generic condition at $\lambda_0$. Then

1. $\mathcal{C}(\lambda_0) = \left\{ \left( \sum_{k=0}^{\infty} \lambda_k \varepsilon^{k/m}, \sum_{k=0}^{\infty} x_k \varepsilon^{k/m} \right) \right\}$;

2. $\lambda_1^m = \langle v_m, \frac{dA}{d\varepsilon}(0) u_1 \rangle = -\frac{\partial}{\partial \varepsilon} \frac{\det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda) = (0, \lambda_0)}}{\partial^{m} \det(\lambda I - A(\varepsilon))|_{(\varepsilon, \lambda) = (0, \lambda_0)}} \neq 0$;

3. $\lambda_1 = \langle v_m, \frac{dA}{d\varepsilon}(0) u_1 \rangle^{1/m}$, $\lambda_2 = \frac{\langle v_{m-1}, \frac{dA}{d\varepsilon}(0) u_1 \rangle + \langle v_m, \frac{dA}{d\varepsilon}(0) u_2 \rangle}{m \lambda_1^{m-2}}$,

$x_0 = u_1$, $x_1 = \lambda_1 u_2$, $x_2 = \left\{ \begin{array}{ll} -\Lambda \frac{dA}{d\varepsilon}(0) u_1 + \lambda_2 u_2, & \text{if } m = 2 \\ \lambda_2 u_2 + \lambda_1^2 u_3, & \text{if } m > 2 \end{array} \right\}$. 
Explicit Recursive Formulas

Notation

- Define the polynomials $p_{j,i} := p_{j,i}(\lambda_1, \ldots, \lambda_{j-i+1})$ in $\lambda_1, \ldots, \lambda_{j-i+1}$, for $j \geq i \geq 0$, as

  $$p_{0,0} := 1,$$
  $$p_{j,0} := 0, \text{ for } j > 0,$$
  $$p_{j,i} := \sum_{s_1 + \cdots + s_i = j} \prod_{q=1}^{i} \lambda_{s_q}, \text{ for } j \geq i > 0$$

- Define the polynomials $r_l := r_l(\lambda_1, \ldots, \lambda_l)$ in $\lambda_1, \ldots, \lambda_l$, for $l \geq 1$, as the expressions

  $$r_1 := 0, r_l := \sum_{s_1 + \cdots + s_m = m+l} \prod_{q=1}^{m} \lambda_{s_q}, \text{ for } l > 1$$
Explicit Recursive Formulas

Theorem (\textit{'09} A. Welters) Explicit Recursive Formulas)

\begin{align*}
\lambda_s &= \begin{cases}
-r_{s-1} + \sum_{i=0}^{s-1} \sum_{j=i}^{s-1} p_{j,i} & m \lambda_1^{m-1} \\
\frac{1}{m} \sum_{k=1}^{m+s-1-j} \frac{d^k A}{d \varepsilon^k} (0) x_{m+s-1-j-km} & , 2 \leq s \leq m
\end{cases} \\
-x_{s-1} + \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} p_{j,i} & m \lambda_1^{m-1} \\
-\sum_{i=0}^{s} p_{s,i} u_{i+1} , & 0 \leq s \leq m - 1
\end{align*}

\begin{align*}
x_s &= \begin{cases}
\sum_{i=0}^{m-1} p_{s,i} u_{i+1} & , s \geq m
\end{cases} \\
\sum_{i=0}^{m-1} p_{s,i} u_{i+1} & - \sum_{j=0}^{s-m} \sum_{k=0}^{j} \sum_{l=1}^{s-j} p_{j,k} \Lambda^{k+1} \frac{d^l A}{d \varepsilon^l} (0) x_{s-j-lm} , & s \geq m
\end{align*}
Local Eigenvalues/Eigenvectors for Analytic Perturbations $A(\varepsilon)$ of $A_0$ → Complete Finite Collection $\mathcal{C}$ of Eigenpair Puiseux Series → for $\lambda_0$ an eigenvalue of $A_0$ the $\lambda_0$-Eigenpair Group $\mathcal{C}(\lambda_0) \neq \emptyset$ ↔ local eigenvalues/eigenvectors of $A(\varepsilon)$, for those eigenvalues near $\lambda_0$

Constructibility: Main Open Problem - Compute via Explicit Recursive Formulas the Puiseux series Coefficients in $\mathcal{C}(\lambda_0)$

Generic Condition ⇒ a Class of Perturbations with a Solution to Open Constructibility Problem

Analytic Perturbation & Generic Condition ≡ Generic Analytic Perturbation of a Single Jordan Block → Explicit Recursive Formulas to Compute the Puiseux series Coefficients in $\mathcal{C}(\lambda_0)$
(‘85) H. Baumgärtel
Analytic Perturbation Theory for Matrices and Operators.

(‘06) A. Figotin and I. Vitebskiy
"Slow Light in Photonic Crystals."

(‘09) A. Welters
"Constructive Perturbation Theory for Matrices with Degenerate Eigenvalues."
<http://arxiv.org/abs/0905.4051>