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joint work with Paolo Buttà
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Eq. (2) would be replaced by the formal equation

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where $\delta_{\partial D}$ is a δ-function, and the function f ("vorticity production") on ∂D is determined by the boundary conditions.

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$$\omega(x,t) := \nabla^{\perp} \cdot u = \partial_{x_1} u_2 - \partial_{x_2} u_1, \qquad \nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1}).$$

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As the boundary is made of two pieces we have

$$f\delta_{\partial\mathcal{C}}(x,t) = f_1(x_1,t)\delta(x_2) + f_2(x_1,t)\delta(x_2-\pi).$$

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We expand $\omega(x,t)$ in the Neumann basis $\{e^{ik_1x_1}\cos k_2x_2\}$:

$$\omega(x,t) = \sum_{k_1\neq 0} \omega_{k_1,0}(t)\, e^{ik_1x_1} + 2\sum_{k_1\in\mathbb{Z}}\sum_{k_2\geq 1} \omega_{k_1,k_2}(t)\, e^{ik_1x_1}\cos(k_2x_2),$$

For convenience we extend ω by parity: $\hat{\omega}_{k_1,k_2} = \omega_{k_1,|k_2|}$, so that

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For convenience we extend ω by parity: $\hat{\omega}_{k_1,k_2} = \omega_{k_1,|k_2|}$, so that

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We obtain an infinite set of coupled ODE's (we drop the hat of $\hat{\omega}_{ij}$). For all $k_1 \in \mathbb{Z},\ k_2 \geq 0$,

$$\dot{\omega}_{k_1,k_2}(t) + N_{k_1,k_2}[\omega(t)] = -k^2 \omega_{k_1,k_2}(t) + f_{\pm,k_1}(t), \tag{7}$$

where $f_{\pm,k_1}(t) = f_{1,k_1}(t) \pm f_{2,k_1}(t)$, the $+$ [resp. $-$] sign is chosen for k_2 even [resp. odd], and

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By applying the inequalities of the previous lemmas we get

$$\begin{aligned}\left|\omega^{(n)}_{k_1,k_2}(t)\right| \quad \leq \quad & \frac{e^{-(1+|k_1|t)/4}}{|k|^\alpha\,(1+|k_1|^\beta)}\Bigg[D_0 + 2\frac{1-e^{-k^2 t/2}}{|k|^{2-\alpha}} \\ & \times\left(C_f D_0 |k_1|^{2-\alpha} + C_f D_2^2 + C_N D_2^2\right)\Bigg]\end{aligned}$$

For $n = 0$ we have $\left\|\omega^{(0)}\right\|_{\alpha,\beta,T} \leq D_0 < D_2$. Proceed by induction and assume that assertion i) holds for any $0 \leq n' < n$ up to some time T_1. As $\omega^{(n-1)}_{k_1,k_2}(0) = \omega_{k_1,k_2}(0)$ for all $n \geq 1$, we have,

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