On Geometric Quantum Confinement in Grushin-type Manifolds

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Quantissima in the Serenissima III

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Quantum Dynamics

Quantum dynamics: Schrödinger equation

$$\mathrm{i}\partial_t \Psi(t) = H\Psi(t)$$

H=self-adjoint operator on $\mathcal{H}$.

Quantum confinement on Ω: $\Psi(t)$ solution to Schrödinger equation

$$\operatorname{supp}\Psi(0) \subset \Omega \;\Rightarrow\; \operatorname{supp}\Psi(t) \subset \Omega, \quad \forall t > 0$$

Geometric Quantum Confinement on Ω: When $\Psi \in L^2(M, \mu)$ with
→ $M = M^- \cup \mathcal{Z} \cup M^+$
→ M^+, M^- Riemannian manifolds

$$\operatorname{supp}\Psi(0) \subset M^{\pm} \;\Rightarrow\; \operatorname{supp}\Psi(t) \subset M^{\pm}\,, \quad \forall t > 0$$

Grushin-type manifold-Grushin cylinder

Example: $M = \mathbb{R} \times \mathbb{S}^1$ with

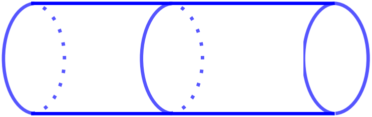
$\rightarrow M^+ = (0, +\infty) \times \mathbb{S}^1$

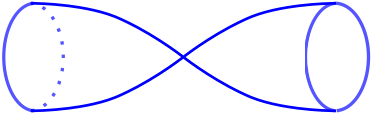
$\rightarrow M^- = (-\infty, 0) \times \mathbb{S}^1$

$\rightarrow \mathcal{Z} = \{0\} \times \mathbb{S}^1$

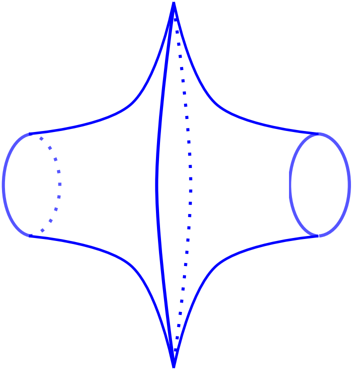
$\rightarrow g_\alpha = \mathrm{d}x^2 + \dfrac{1}{|x|^{2\alpha}}\mathrm{d}y^2.$

$\alpha = 0$





$\alpha > 0$



$\alpha < 0$

Geometric Quantum Confinement

Geometric Quantum Confinement on Ω: When $\Psi \in L^2(M, \mu)$ with

$\to$ $M = M^- \cup \mathcal{Z} \cup M^+$

$\to$ M^+, M^- Riemannian manifolds

$$\operatorname{supp} \Psi(0) \subset M^\pm \;\Rightarrow\; \operatorname{supp} \Psi(t) \subset M^\pm\,, \quad \forall t > 0$$

when $\Psi(t)$ is evolved **with the flow of any self-adjoint extension of the minimal Laplacian.**

Literature:

[1] Boscain and Prandi, JDE (2016)

[2] Prandi, Rizzi and Seri, arXiv (2016)

[3] Posilicano, Arab. Journal of Mathematics (2014)

[4] G., Michelangeli, Pozzoli, (to appear in ZAMP)

Geometric Quantum Confinement in the Grushin-plane

Operator:

$$H = \mathrm{div}_{\mu_\alpha} \circ \nabla$$

Laplace-Beltrami operator.

Manifold: Grushin-type plane: $\alpha \in [0, +\infty)$

$$M := \left(\mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+\right) \times \mathbb{R} \qquad g_\alpha := \mathrm{d}x \otimes \mathrm{d}x + \frac{1}{|x|^{2\alpha}} \mathrm{d}y \otimes \mathrm{d}y$$

Metric induces a measure

$$\mathrm{d}\mu_\alpha := \sqrt{\det g_\alpha}\, \mathrm{d}x \wedge \mathrm{d}y = \frac{1}{|x|^\alpha} \mathrm{d}x \wedge \mathrm{d}y$$

Hilbert space in quantum mechanics

$$\mathcal{H}_\alpha = L^2(M, \mathrm{d}\mu_\alpha)$$

The problem is equivalent to the essential self-adjointness of H on $C_0^\infty(M \setminus \mathcal{Z})$

Main result

Theorem (G. - Pozzoli - Michelangeli, 2019)

The operator H

$$H = -\frac{\partial^2}{\partial x^2} - |x|^{2\alpha}\frac{\partial^2}{\partial y^2} + \frac{\alpha}{x}\frac{\partial}{\partial x}$$

a) is essentially self adjoint for $\alpha \in [1, +\infty)$; $\rightarrow$ **Confinement**

b) is not essentially self-adjoint for $\alpha \in [0, 1)$.$\rightarrow$ **No Conf**

on $L^2(M, \mathrm{d}\mu_\alpha)$.

Strategy of the proof

→ Unitary equivalent problem on

$$UL^2(M, \mathrm{d}\mu_\alpha) = \int_{\mathbb{R}}^{\oplus} \Big(L^2(\mathbb{R}^-, \mathrm{d}r) \oplus L^2(\mathbb{R}^+, \mathrm{d}r) \Big) \, \mathrm{d}\xi$$

and on each fibre

$$UHU^* \restriction_\xi = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + |r|^{2\alpha}\xi^2 + \frac{\alpha(\alpha+2)}{4r^2}$$

→ Essentially self-adjointness

→ (our technical result)

$$H \text{ ess. s.a.} \Leftrightarrow UHU^* \restriction_\xi \text{ is ess. s.a. for any } \xi$$

Thank you for your attention