6 Problems

In what follows, we denote by X a Polish space, by $C_b(X)$ the space of bounded continuous functions with the supremum norm, by $\mathcal{B}_X$ the Borel σ-algebra on X, and by $\mathcal{P}(X)$ the space of probability measures on $(X, \mathcal{B}_X)$ endowed with the weak topology. We write $\langle V, \mu\rangle$ for the integral of $V \in C_b(X)$ against the measure $\mu \in \mathcal{P}(X)$.

6.1 Formulations

Problem 1. Let $\lambda \in \mathcal{P}(X)$. Prove that the function $Q : C_b(X) \to \mathbb{R}$ taking V to $\ln\langle e^V, \lambda\rangle$ is 1-Lipschitz and convex.

Problem 2. Let $\lambda \in \mathcal{P}(X)$. Prove that the function $\mu \mapsto \mathrm{Ent}(\mu \,|\, \lambda)$ acting from $\mathcal{P}(X)$ to $\mathbb{R}_+ \cup \{+\infty\}$ is lower semicontinuous and strictly convex.

Problem 3. For $\lambda \in \mathcal{P}(X)$ and $A \geq 0$, let $M_A := \{\mu \in \mathcal{P}(X) : \mathrm{Ent}(\mu \,|\, \lambda) \leq A\}$. Prove that $M_A \subset \mathcal{P}(X)$ is compact.

Problem 4. Using the notation of Problems 1 and 2, prove that, for any $V \in C_b(X)$ and $\lambda, \mu \in \mathcal{P}(X)$, we have

$$\mathrm{Ent}(\mu \,|\, \lambda) = \sup_{V \in C_b(X)} \big(\langle V, \mu\rangle - Q(V)\big), \tag{6.1}$$

$$Q(V) = \sup_{\mu \in \mathcal{P}(X)} \big(\langle V, \mu\rangle - \mathrm{Ent}(\mu \,|\, \lambda)\big). \tag{6.2}$$

Problem 5. Prove that the supremum in (6.2) is saturated on the unique measure $\mu_V = Z_V^{-1} e^V \lambda$, where $Z_V = \langle e^V, \lambda\rangle$, whereas the supremum in (6.1) may not be attained.

Problem 6. Let $V \in C_b(X)$ and let $\lambda \in \mathcal{P}(X)$. Prove the *contraction relation*

$$\inf\big\{\mathrm{Ent}(\mu \,|\, \lambda) : \langle V, \mu\rangle = r\big\} = \sup_{\alpha \in \mathbb{R}} (r\alpha - \ln Z_{\alpha V}). \tag{6.3}$$

Problem 7. Let $\lambda \in \mathcal{P}(X)$, let $V \in C_b(X)$ be a function that is not λ-almost everywhere constant, and let $I_V(r)$ be the expression on the right-hand side of (6.3). Prove the following properties.

(a) Let $q(\alpha) = \ln Z_{\alpha V}$. Then $q \in C^\infty(\mathbb{R})$ and q' is strictly increasing.

(b) Let us denote by S the support of λ and define the numbers $A = \inf_S V$ and $B = \sup_S V$. Then $q'(\alpha) \to A$ as $\alpha \to -\infty$ and $q'(\alpha) \to B$ as $\alpha \to +\infty$.

(c) The function $I_V(r)$ is finite for any $r \in (A, B)$.

Problem 8. Let λ and V be as in Problem 7, let $Z_\beta = \langle e^{-\beta V}, \lambda\rangle$, and let μ_β be the measure $Z_\beta^{-1} \mu_{-\beta V}$. Prove the following properties.

(a) For any $r \in (A, B)$, there is a unique $\beta \in \mathbb{R}$ such that the measure μ_β is the unique minimiser of the infimum on the left-hand side of (6.3)

(b) For any $\beta \in \mathbb{R}$, we have

$$-\ln Z_\beta = \inf_{\mu \in \mathcal{P}(X)} \big(\beta\langle V, \mu\rangle + \mathrm{Ent}(\mu \,|\, \lambda)\big). \tag{6.4}$$

Moreover, the infimum on the right-hand side is attained only at $\mu = \mu_\beta$.

Problem 9. Let $\lambda \in \mathcal{P}(X)$ and $V \in C_b(X)$ be such that V is not λ-almost everywhere constant, and let numbers A and B be defined in Part (b) of Exercise 7. Suppose a measure $\mu \in \mathcal{P}(X)$ be such that $\mathrm{Ent}(\mu \,|\, \lambda) < \infty$ and $\langle V, \mu\rangle \in (A, B)$. Prove that there positive numbers ε_0 and C such that, for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, one can find a measure $\mu_\varepsilon \in \mathcal{P}(X)$ satisfying the relations

$$\langle V, \mu_\varepsilon\rangle = \langle V, \mu\rangle + \varepsilon, \quad \mathrm{Ent}(\mu_\varepsilon \,|\, \lambda) \le \mathrm{Ent}(\mu \,|\, \lambda) + C\,|\varepsilon|. \tag{6.5}$$

Problem 10. Let $(\mu_n)_{n\ge 1}$ be a sequence of probability measures on X. Define a set function $S : \mathcal{B}_X \to [0, 1]$ by the relation

$$S(\Gamma) = \limsup_{n\to\infty} \big(\mu_n(\Gamma)\big)^{1/n}, \quad \Gamma \in \mathcal{B}_X. \tag{6.6}$$

Prove that S is a Ruelle–Lanford function.

Problem 11. Let X be a topological vector space and let $I : X \to [0, +\infty]$ be a lower semicontinuous function such that

$$I\bigg(\frac{1}{2}(x_1 + x_2)\bigg) \le \frac{1}{2} I(x_1) + \frac{1}{2} I(x_2) \quad \text{for any } x_1, x_2 \in X. \tag{6.7}$$

Prove that I is convex.

Problem 12. Prove Sanov's theorem for an i.i.d. sequence of random variables in a compact metric space, using the Ruelle–Lanford approach and the result on decoupled measures.