## §3. TDL via LDP

- **Sanov theorem**

$X$ Polish space, $\mathcal{B}(X)$, $\mathcal{P}(X)$, $(\xi_k)_{k\geq 1}$ i.i.d. r.v. in $X$, $\underset{\mathcal{P}(X)}{\underset{\cap}{\lambda}} = \mathcal{L}(\xi_1)$

Empirical measures: $\quad \nu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{\xi_k}$

$$\mu \in \mathcal{P}(X), \quad \underset{I(\mu)}{\underset{\shortparallel}{\mathrm{Ent}(\mu|\lambda)}} = \int_X \ln \frac{d\mu}{d\lambda}\, d\mu \geq 0$$

**Theorem** $\quad \forall \Gamma \in \mathcal{B}(\mathcal{P}(X))$

$$-I(\mathring{\Gamma}) \leq \liminf_{n\to\infty} \frac{1}{n}\ln \mathbb{P}\{\nu_n \in \Gamma\} \leq \limsup_{n\to\infty} \frac{1}{n}\ln \mathbb{P}\{\nu_n\in\Gamma\} \leq -I(\overline{\Gamma})$$

where $\mathring{\Gamma}/\overline{\Gamma}$ interior/closure of $\Gamma$, $I(A) = \inf_A I$.

__Dictionary__

| | SM | LDP |
|---|---|---|
| $\mathrm{Ent}(\mu\vert\lambda)$ | Entropy | Rate function |
| $I(\mu) = \sup_{V\in C_b(X)} \big(\underset{\int_X V d\mu}{\underset{\shortparallel}{\langle V,\mu\rangle}} - \ln Z_V\big)$ <br> $\ln Z_V = \sup_{\mu\in\mathcal{P}(X)} \big(\langle V,\mu\rangle - I(\mu)\big)$ | variational principles | Legendre transform / Fenchel |
| $Z_V = \langle e^V, \lambda\rangle$ | partition function | $\ln Z_V$ = pressure |

$$V = -\beta H, \quad Z_V = \langle e^{-\beta H}, \lambda\rangle = Z_\beta, \ \beta>0$$

$$\underbrace{-\frac{1}{\beta}\ln Z_V}_{\text{free energy}} = \inf_{\mu\in\mathcal{P}(X)} \Big(\underbrace{\langle H,\mu\rangle}_{\text{total energy}} + \underbrace{\frac{1}{\beta}\mathrm{Ent}(\mu|\lambda)}_{-TE}\Big)$$

## Maximum entropy principle

**Theorem** $\mathscr{C} \subset \mathcal{P}(X)$ closed subset s.t. $I(\mathring{\mathscr{C}}) = I(\mathscr{C}) < \infty$. Then

(a) $\exists!\ \bar\mu \in \mathcal{P}(X)$ $\quad I(\bar\mu) = I(\mathscr{C})$.

(b) $\mathbb{P}\{\vartheta_n \in \cdot \mid \vartheta_n \in \mathscr{C}\} \rightharpoonup \delta_{\bar\mu}$

$G \subset \mathcal{P}(X)$ open set, $G \not\ni \bar\mu \Rightarrow \exists\, b > 0\ \mathbb{P}\{\vartheta_n \in G \mid \vartheta_n \in \mathscr{C}\} \le e^{-bn}$, $n \gg 1$

(c) $\forall V \in C_b(X)$ $\forall k \in \mathbb{N}$

$$\lim_{n\to\infty} \mathbb{E}\{V(\xi_k) \mid \vartheta_n \in \mathscr{C}\} = \langle V, \bar\mu\rangle$$

**Proof of (c)** $(\xi_1, \dots, \xi_n) \to (\xi_{i_1}, \dots, \xi_{i_n}) \Rightarrow \mathbb{E}\{V(\xi_k) \mid \vartheta_n \in \mathscr{C}\} = \mathbb{E}\{V(\xi_\ell) \mid \vartheta_n \in \mathscr{C}\}$

$$\mathbb{E}\{V(\xi_k) \mid \vartheta_n \in \mathscr{C}\} = \frac{1}{n}\sum_{\ell=1}^{n} \mathbb{E}\{V(\xi_\ell) \mid \vartheta_n \in \mathscr{C}\}$$

$$= \mathbb{E}\Big\{\underbrace{\frac{1}{n}\sum_{\ell=1}^{n} V(\xi_\ell)}_{\langle V, \vartheta_n\rangle} \,\Big|\, \vartheta_n \in \mathscr{C}\Big\} \xrightarrow[n\to\infty]{} \langle V, \bar\mu\rangle$$

$\mu \mapsto \langle V, \mu\rangle$

$\mathcal{P}(X) \to \mathbb{R}$

$\blacksquare$

- ## TDL for an ideal gas (§A4, Lanford)

$X \subset \mathbb{R}^3$, velocities are distributed according to a law $\lambda \in \mathcal{P}(X)$

$V \in C_b(X)$, $\xi_1, \dots, \xi_n$ velocities, i.i.d with law $\lambda$

$$\lim_{n\to\infty} \mathbb{P}\Big\{\xi_k \in \cdot \,\Big|\, \frac{1}{n}\sum_{\ell=1}^{n} V(\xi_\ell) = r\Big\} \quad ? \qquad \vartheta_n = \frac{1}{n}\sum_{\ell=1}^{n} \delta_{\xi_\ell}$$

**Theorem** Let $\lambda \in \mathcal{P}(X)$, $V \in C_b(X)$, $S = \operatorname{supp} \lambda$,

$$A := \inf_S V < \sup_S V =: B$$

Then:

(a) $\forall r \in (A, B) \quad \exists!\, \beta \in \mathbb{R}$ such that

$$\langle V, \mu_\beta \rangle = r, \quad \mu_\beta = \frac{e^{-\beta V}}{Z_\beta} \lambda, \quad Z_\beta = \langle e^{-\beta V}, \lambda \rangle.$$

(b) $\forall k \geq 1$

$$\lim_{\delta \to 0^+} \lim_{n \to \infty} \mathbb{P}\{ \xi_k \in \cdot \mid |\langle V, \nu_n \rangle - r| \leq \delta \} = \mu_\beta$$

Proof 1) <u>Existence of $\beta$</u>

$$q(\alpha) = \ln Z_{\alpha V} = \ln \langle e^{\alpha V}, \lambda \rangle \Rightarrow q'(\alpha) = \langle V, \mu_{\alpha V} \rangle, \quad \mu_{\alpha V} = \frac{e^{\alpha V}}{Z_{\alpha V}} \lambda$$

$$q'(\alpha) = \frac{\langle V e^{\alpha V}, \lambda \rangle}{\langle e^{\alpha V}, \lambda \rangle} = \left\langle \frac{V e^{\alpha V}}{Z_{\alpha V}}, \lambda \right\rangle = \langle V, \underbrace{\frac{e^{\alpha V}}{Z_{\alpha V}} \lambda}_{\mu_\beta \text{ for } \alpha = -\beta} \rangle$$

$$\boxed{q'(\alpha) = r}$$

$$\lim_{\alpha \to -\infty} q'(\alpha) = A, \quad \lim_{\alpha \to +\infty} q'(\alpha) = B$$

$$q''(\alpha) = \langle V^2, \mu_{\alpha V} \rangle - \langle V, \mu_{\alpha V} \rangle^2 > 0$$

2) <u>Limit as $n \to +\infty$</u>

$$\mathcal{C}_\delta = \{ \mu \in P(\mathcal{X}) : |\langle V, \mu \rangle - r| \leq \delta \} \quad \text{closed subset}$$

$$I(\dot{\mathcal{C}}_\delta) = I(\mathcal{C}_\delta), \quad m(\delta) = \inf_{\mu \in \mathcal{C}_\delta} \mathrm{Ent}(\mu \mid \lambda), \quad \delta \geq 0$$

**Lemma** The function $\mathbb{R}_+ \ni \delta \to m(\delta) \in \mathbb{R}_+$ is non-decreasing and left-continuous.

$$0 \leq \delta' < \delta \qquad \underset{\inf_{\mu \in \mathcal{C}_{\delta'}} I(\mu)}{\overset{\shortparallel}{m(\delta')}} \geq \inf_{\mu \in \dot{\mathcal{C}}_\delta} I(\mu) \underset{I(\dot{\mathcal{C}}_\delta)}{=} \geq m(\delta) \Rightarrow \lim_{\delta' \uparrow \delta} m(\delta') = m(\delta)$$

$\forall \delta \geq 0 \quad \exists \bar{\mu}_\delta \in \mathcal{C}_\delta \quad$ s.t. $\quad I(\bar{\mu}_\delta) = I(\mathcal{C}_\delta)$

$\forall k \geq 1 \quad \forall \delta > 0 \quad \lim_{n\to\infty} \mathbb{P}\{\xi_k \in \cdot \mid \nu_n \in \mathcal{C}_\delta\} = \bar{\mu}_\delta$

3) Limit as $\delta \to 0$

$$S_n = \frac{1}{n}\sum_{\ell=1}^{n} V(\xi_\ell), \quad \nu_n = \frac{1}{n}\sum_{\ell=1}^{n} \delta_{\xi_\ell}; \ \langle V, \nu_n\rangle = S_n$$

$$I_V(r) = \inf\{I(\mu) : \langle V, \mu\rangle = r\}$$

$$\inf \emptyset = +\infty$$

$$r \in (A, B), \quad I_V(r) = \sup_{\alpha\in\mathbb{R}} (r\alpha - q_V(\alpha)), \quad q_V(\alpha) = \ln\langle e^{\alpha V}, \lambda\rangle$$

$$+\infty > I_V(r) = I(\mathcal{C}_0) = m(0) \geq m(\delta) = I(\mathcal{C}_\delta) = I(\bar{\mu}_\delta)$$

It remains to prove that $\bar{\mu}_\delta \to \mu_\beta$, $\langle V, \mu_\beta\rangle = r$

- $\{\bar{\mu}_\delta\} \subset \mathcal{P}(X)$ is relatively compact

  $\forall C > 0 \quad \{\mu \in \mathcal{P}(X) : \mathrm{Ent}(\mu \mid \lambda) \leq C\} \subset\subset \mathcal{P}(X)$

- $\exists \delta_k \downarrow 0 \quad \bar{\mu}_{\delta_k} \to \bar{\mu}$; if we prove that $\bar{\mu} = \mu_\beta$, then the result will follow.

$I(\bar{\mu}_\delta) = I(\mathcal{C}_\delta)$

Claim: $\boxed{I(\bar{\mu}) = I(\mathcal{C}_0)}$

$$I(\bar{\mu}) \leq \liminf_{k\to\infty} I(\bar{\mu}_{\delta_k}) = \liminf_{k\to\infty} I(\mathcal{C}_{\delta_k}) \leq I(\mathcal{C}_0)$$

$\bar{\mu} \in \mathcal{C}_0$

$$I(\bar{\mu}) = \inf\{\mu \in \mathcal{P}(X) : \langle V, \mu\rangle = r\} = I_V(r)$$

$$= \sup_{\alpha\in\mathbb{R}} (r\alpha - q_V(\alpha)) = -r\beta - q_V(-\beta)$$

$q_V'(-\beta) = r \in (A, B)$

$$= \langle -\beta V, \bar{\mu}\rangle - \ln Z_\beta$$

$$\ln Z_\beta = \sup_{\mu\in\mathcal{P}(X)} (\langle -\beta V, \mu\rangle - \mathrm{Ent}(\mu \mid \lambda)), \quad \mu_\beta = \frac{e^{-\beta V}}{Z_\beta}\lambda$$

$$\ln Z_\beta = \langle -\beta V, \mu_\beta \rangle - \mathrm{Ent}(\mu_\beta | \lambda)$$

By uniqueness, we must have $\bar{\mu} = \mu_\beta$ ∎

$\dot{x} = V(x) + \sum_{j=1}^{n} V_j(x)\dot{\beta}_j$, $\{x_t\}$ solutions; this is a Markov process

$$\nu_t = \frac{1}{t}\int_0^t \delta_{x_s}\, ds \longrightarrow \langle V, \nu_t \rangle = \frac{1}{t}\int_0^t V(x_s)\, ds \longrightarrow \langle V, \lambda \rangle$$

$$\downarrow$$

$$\delta_\lambda$$

$$\bar{\nu}_t = \frac{1}{t}\int_0^t \delta_{(x_r,\, r \geq s)}\, ds$$

$$\Uparrow$$

$$\mathcal{P}(X^{\mathbb{R}_+})$$

$(x_n)$, $X$

- $\exists T \geq 1$ $\quad P_T(x,\cdot) \sim P_T(y,\cdot)$ $\quad \forall x,y \in X$
- $\forall \delta > 0\ \exists T > 0\ \exists p > 0$ $\quad \forall x,y \in X$ $\quad P_T(x, B(y,\delta)) \geq p$

$\Big\} \Rightarrow$ LDP

Problem 4 $\quad \boxed{\mathrm{Ent}(\mu|\lambda) = \sup_{V \in C_b(X)} \left(\langle V, \mu \rangle - Q(V)\right),}$ $\quad Q(V) = \ln Z_V = \langle e^V, \lambda \rangle$

$$Q(\theta V + (1-\theta)W) = \ln \int e^{\theta V + (1-\theta)W}\, d\lambda \qquad \frac{1}{\theta}, \frac{1}{1-\theta}$$

$$= \ln \int \cdots$$

$$\boxed{Q(V) = \sup_{\mu \in \mathcal{P}(X)} \left(\langle V, \mu \rangle - \mathrm{Ent}(\mu|\lambda)\right)}$$

$X$ compact metric space $\qquad C_b(X) = C(X)$, Riesz–Markov, $C(X)^* = \mathcal{M}(X)$

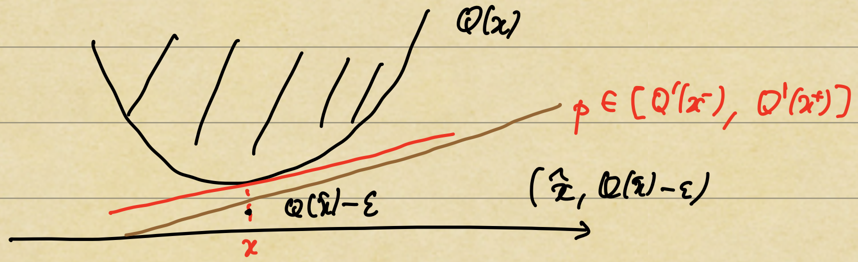
$I(\mu)$ = Legendre transform of $Q(V)$, $\quad Q: C(X) \to \mathbb{R}$

$\qquad I: \mathcal{M}(X) \to \mathbb{R} \cup \{+\infty\}$

$\mathcal{X}$, $\mathcal{Y} = \mathcal{X}^*$ $\qquad Q: \mathcal{X} \to \mathbb{R}$, $\quad I = Q^*: \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$

$I^* = Q$

1) $\mathcal{X} = \mathbb{R}$




$$I(p) = \sup_{x \in \mathbb{R}} (px - Q(x)) \geq px - Q(x) \quad \forall x \in \mathbb{R} \Rightarrow Q(x) \geq px - I(p)$$

$\forall p \ \forall x$

$$\Rightarrow Q(x) \geq \sup_{p \in \mathbb{R}} (px - I(p)) = I^*(x)$$

$Q(x) \geq I^*(x)$; $\qquad Q(x) \leq I^*(x)$

$\hat{x}$, $\varepsilon > 0$: $\qquad Q(\hat{x}) - \varepsilon \leq I^*(\hat{x})$

$\exists\, p \in \mathbb{R},\ c \in \mathbb{R}$ $\qquad Q(\hat{x}) - \varepsilon \leq p\hat{x} - c$

$\qquad Q(x) \geq px - c \quad \forall x \in \mathbb{R} \qquad -c \leq Q(x) - px$

$$Q(\hat{x}) - \varepsilon \leq p\hat{x} - c \leq p\hat{x} + \underbrace{Q(x) - px} \quad \forall x \in \mathbb{R}$$

$$Q(\hat{x}) - \varepsilon \leq p\hat{x} - I(p) \leq I^*(\hat{x})$$

2) Geometric version of Hahn-Banach

$Q : \mathcal{X} \to \mathbb{R}$ convex function

$\forall x \in \mathcal{X} \qquad Q(x) = \sup\{\langle p, x\rangle - c : \ \forall x' \in \mathcal{X} \ \langle p, x'\rangle - c \leq Q(x')\}$

$\hat{x} \in \mathcal{X}$, $\varepsilon > 0 \to \exists p \in \mathcal{X}^* = \mathcal{Y} \ \exists c \in \mathbb{R}$

$\qquad \forall x \in \mathcal{X} \quad \langle p, x\rangle - c \leq Q(x)$

$\qquad Q(\hat{x}) - \varepsilon \leq \langle p, \hat{x}\rangle - c$ ∎