From Classical Physics to the *Miracles of the Quantum World*[1]

"I was born not knowing and have only had a little time to change that here and there" (R.P. Feynman)

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Overview of Lectures

1. The Passage from Classical Physics to Quantum Physics - a Leisurely Introduction

It seems clear that the present quantum mechanics is not in its final form. (P.A.M. Dirac)

I present a short account of the passage from classical physics to quantum physics – from the Platonic Realm, where strict causality, determinism and reversibility prevail, to the Aristotelian Realm, where chance occupies center stage, the future is uncertain and the flow of time is irreversible. This passage represents a revolution not only in our conception of the paradigms underlying natural science and our description and manipulation of Nature, but also in the area of new technologies born from quantum science, such as lasers, semi-conductors, transistors (and their many applications, e.g. in computers), superconductors, nuclear magnetic resonance imaging, nuclear power plants, atomic weapons, ...

2. The Classical No-Go Theorems: Kochen-Specker and Bell

As my main task in this talk, I will attempt to explain to you the Kochen-Specker Theorem, which says that there does not exist a hidden variables theory reproducing the contents of Quantum Mechanics (QM), and Bell's Inequalities. The mathematics underlying the Kochen-Specker theorem is related to Gleason's theorem. I will mention an extension of Gleason's theorem to general von Neumann algebras.

3. The Inadequacy of the Schrödinger Equation - Wigner's Friend and How to Get Beyond it in the "ETH-Approach to Quantum Mechanics."

The purpose of this lecture is to extend the standard formalism of QM and complete it (Dirac!) in such a way that the resulting theory makes sense. The extension, yielding a new Law of Nature, is called "ETH - Approach to QM."
The ETH - Approach to QM supplies the fourth one of four pillars QM rests upon:
(i) Physical quantities characteristic of a physical system are represented by s.-a. linear operators.
(ii) The time evolution of operators representing physical quantities is given by the Heisenberg equations;
(iii) Introduction of meaningful notions of Potential and Actual Events and of states.
(iv) Proposal of a general statistical Law for the Time Evolution of states.
Core of lecture: Besides sketching the ETH-Approach to QM, I will discuss simple models of a very heavy atom coupled to the radiation field in a limit where the speed of light tends to ∞, illustrating the ETH-Approach.
General goal: I am determined to remove some of the enormous confusion befuddling many colleagues who claim to work on the foundations of QM... Of course, hardly anybody expects that I will succeed – but I do!

4. ETH-Approach to Quantum Mechanics - Non-Relativistic and Relativistic

I will explain why neither (classical) Relativistic Theories, nor Quantum Theory enable one to predict the future with certainty. I will then sketch why "Einstein causality", or locality, is an essential property of relativistic Quantum Theories. I will then continue to present the "ETH approach" to Quantum Theory – for non-relativistic Quantum Mechanics and, to conclude, for relativistic Quantum Theory.

5. Indirect Measurements in Quantum Mechanics - From Haroche-Raymond to Darwin & Mott

I will study the effective quantum dynamics of systems under repeated observation, more specifically ones interacting with a chain of independent probes (such as photons, neutrons, atoms, ...) which, afterwards, are subject to a projective measurement and are then lost. This leads to a theory of indirect measurements of time-independent quantities (non-demolition measurements). Subsequently, a theory of indirect weak measurements of time-dependent quantities is outlined, and a new family of diffusion processes, dubbed quantum jump processes, is described.

Contents of Introductory Lectures, Acknowledgements

1. Summary of Lectures 1 - 3
2. An Impressionistic Account of Classical Physics
3. The *"Dada"* of Quantum Mechanics
4. Time, Events and States in *QM*

Acknowledgements:
I wish to thank my collaborators, and in particular my last PhD student *Baptiste Schubnel* and my friends *Philippe Blanchard, Alessandro Pizzo* and *Gang Zhou* for the joy of joint efforts, as well as many colleagues at numerous institutions for instructive (and sometimes rather controversial) discussions on *QM*. Some of the ideas presented in the following are similar to ideas of the late *Rudolf Haag* and have been influenced by results due to *Detlev Buchholz*.

1. Summary of Lectures 1 - 3

"It seems clear that the present quantum mechanics is not in its final form." (P.A.M. Dirac)

My goal in these lectures is to sketch a possible **completion** of QM.

Summary:

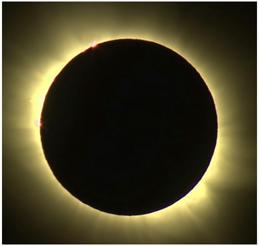
I will first present a short account of the transition from *classical physics* to *quantum physics* – from the Platonic Realm, where strict **causality, determinism** and **reversibility** prevail, to the Aristotelian Realm, where **chance** occupies center stage, the future is uncertain and **time has an arrow**. This transition represents a revolution not only in our conception of the paradigms underlying natural science and our description and manipulation of Nature, but also in the area of new technologies born from quantum science, such as *lasers, semi-condurctors, transistors (and their many applications, e.g. in computers), superconductors, nuclear magnetic resonance imaging, nuclear power plants, atomic waepons, ...*

I then present a leisurely introduction to Quantum Theory, summarizing various general (and possibly puzzling) facts. I conclude with a preview of the *"ETH- Approach"* to Quantum Mechanics discussed in some detail in subsequent lectures.

2. An Impressionistic Account of Classical Physics

Classical Physics was born from observations of the night sky (observatories in Samarkand, Jaipur, ...), and in particular of the motion of the moon and the planets, of solar and lunar eclipses, moreover from the study of optics (reflection and refraction of light – *Snellius'* laws) and of simple (hydro-) static phenomena (*Archimedes'* lever rule and law of buoyancy).

Images of the night sky, a solar eclipse, and of the Jaipur observatory:



Paradigms of Classical Physics

The astronomers and philosophers of antiquity discovered that there are regularities, *Laws*, which rule the heavenly phenomena. Their discoveries inspired the pre-socratic natural philosopher *Leucippus* and his pupil *Democritus* to formulate the following paradigms, which, to this day, contine to influence scientific thinking enormously:

I. The natural phenomena are ruled by rigid eternal laws

As *Socrates* taught his followers: The Universe is ordered and governed by a wonderful intelligence and superior wisdom.

II. The Law of Causality

Every event is the necessary consequence of some cause.

III. Matter is composed of atoms

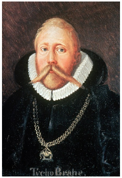
The smallest constituents of matter are atoms, which are indecomposable; ($\nearrow$ *Einstein, Perrin*). There exist finitely many species of atoms (in antiquity sometimes identified with the Platonic solids).

In classical mechanics, atoms are often idealised as *point particles.*

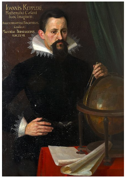
From the observed motion of planets to Kepler's Laws and Newtonian Mechanics

Already in antiquity, the geocentric system of *Ptolemy* (Earth in center of Universe, orbits of planets described by epi-cycles) was abandoned in favor of the heliocentric system (Sun in center of Universe) propagated later by *Copernicus*. And *Aristotle* proposed foundations of mechanics, which, however, failed.

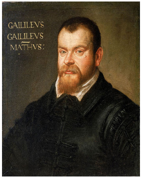
In the $16^{th}/17^{th}$ Century, the precise observations of planetary motions by *Tycho Brahe* gave rise to *Kepler's Laws* (1609) (orbits of planets= ellipses with the sun in a focus, area law,...). These laws and *Galileo's* discovery of the importance of the notion of *acceleration* evntually gave rise to the birth of *Newtonian mechanics*.



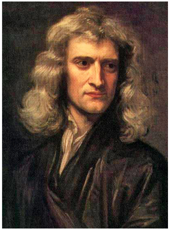
Brahe



Kepler



Galileo



Newton

The notion of *"state"* in Newtonian mechanics

The eminent French Mathematician *René Thom* insisted on the idea that mathematical notions (e.g., function) and concepts (e.g., smoothness) play a crucial role in the discovery of physical laws. (Of course, theor. ideas and concepts play an essential role in planning experiments!)

Besides the notions of *function, curve, acceleration, ...*, an example of a mathematical notion essential in the development of Newtonian mechanics is the notion of **state**. In Newtonian mechanics, the *state of the Universe* is described by

$$\xi = \left(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N\right),$$

specifying the positions, $\mathbf{x}_j$, in space and the velocities, $\mathbf{v}_j$, of all heavenly bodies, $j = 1, 2, \ldots, N$. Alas, N is an unknown, giant number!

Obviously we don't know the positions and velocities of **all** heavenly bodies! However, the solar system is far away from other stars and planetary systems, whose influence on motions in the solar system we can therefore neglect at first. And the influence of small bodies, such as asteroids and comets, on planetary motion is most often neglected, too.

$\rightarrow$ *Physics only works when intelligent idealizations and approximations are made!*

Newton's Equations of Motion

Thus, let us consider a mechanical system, *S*, e.g., the solar system, idealized as an **isolated system** of relatively few "particles". The set of all states of *S* is its state space, $\mathfrak{X}$.

To formulate **dynamical laws**, the notion of *time* is crucial. For *Newton*, *time* takes real values and is absolute. The state of *S* at time t is given by a t-dependent point $\xi_t = \big(\mathbf{x}_1(t), \mathbf{v}_1(t), \ldots, \mathbf{x}_N(t), \mathbf{v}_N(t)\big)$ in $\mathfrak{X}$.

In his **magnum opus** *'Philosophiæ Naturalis Principia Mathematica'*, Newton proposed a dynamical law for the time-dependence of the state ξ_t in the form of a system of ordinary differential equations:

$$\frac{d\xi_t}{dt} \equiv \dot{\xi}_t = X(\xi_t), \qquad \xi_t \in \mathfrak{X}\,, \tag{1}$$

where X is a map from $\mathfrak{X}$ to the "space" of vectors attached to $\mathfrak{X}$; (in mathematics, X is called a vector field over $\mathfrak{X}$).

Puzzle: The sun and the planets are *extended bodies.* Why is it reasonable to describe the state of the solar system by merely specifying the *center-of-mass* positions and -velocities of the sun and the planets? The reason for this lies in the form of the law for the gravitaional force.

Newton's Inverse-Square Law and Newton's Theorem

The gravitational force, $\vec{F}$, between two point-like bodies of masses m and M separated by a distance r is given by

$$\vec{F} = -G_N \frac{m \cdot M}{r^3} \vec{r} \qquad (G_N : \text{Newton's const.}) \tag{2}$$

Newton's Theorem: *The gravitational force exerted by a spherically symmetric body of mass M on bodies outside it is identical to the force exerted by a point-like body of the* **same** *mass M concentrated in its center of mass.* – (This took Newton years to prove.)

It is a good approximation to describe the sun and the planets as spherically symmetric bodies. Using Eqs. (1) and (2) and Newton's Theorem for two bodies ($N = 2$), one derives the three Laws of *Kepler*, as *Newton* demonstrated.

When $N \geq 3$ the mathematical problems encountered in the analysis of Eqs. (1) and (2) become formidable! Great mathematicians including *Émilie (Marquise) du Châtelet, Euler, Maupertuis, Lagrange, Laplace, Hamilton* and *Poincaré* have made monumental contributions to the study of Newtonian mechanics (including, e.g., discovery of chaos for $N \geq 3$!)

Physical Quantities ("Observables") in classical mechanics

In physics, one attempts to characterize an arbitrary physical system, S, by specifying a family, $\mathcal{O}_S$, of **physical quantities** whose values can, in principle, be determined precisely in every state $\xi \in \mathfrak{X}$ of S. Among physical quantities there are: The total energy of S, its total mass, the number and positions of all particles of S that are found in a region Ω of physical space, ...

All physical quantities of S are represented by *real-valued functions* on the state space, $\mathfrak{X}$, of S. They have the following properties:

- All physical quantities in $\mathcal{O}_S$ take precise values in every (pure) state $\xi \in \mathfrak{X}$.
- (Provided that the force law X in Eq. (1) has suitable properties) one has that if ξ_{t_0} is known at a single time t_0 one can in principle calculate the values of all elements of $\mathcal{O}_S$ at **all** times t!
- Paradigms **I** and **II** of Leucippus and Democritus are valid; and **III** is considered to be an excellent idealization in celestial mechanics.
- The world as described by Newtonian (Hamiltonian) mechanics resembles an ultra-reliable *Swiss watch* – a feature that made Newton tumble into a nervous depression ...

The Problem of Scales in Physics

The project of solving Newton's equations of motion for a system consisting of $N = \mathcal{O}(10^{23})$ particles, e.g., all atoms in a mole of gas, is bound to fail. There isn't any computer that could be used to implement it. – Even if there were such a computer we would not learn anything comprehensible from the quasi-infinity of numerical data it would produce!

One must therefore construct **approximate descriptions**, or *Effective Theories*, of systems containing a very large number of particles, such as a galaxy or a gas. This is accomplished by introducing *coarse graining* (collective degrees of freedom). – Among such efffective theories are: **Thermodynamics** (Carnot, Mayer, Clausius, Lord Kelvin,...) & **Stat. Mechanics** (Maxwell, Boltzmann, Einstein, Gibbs,...), **Continuum Mechanics** & **Fluid Dynamics** (Bernoulli, Euler, Navier, Stokes, Cauchy, Helmholtz, Kolmogorov,...), **Density Functional Theory**, etc.

Such *Effective Theories* are used to describe phenomena on *macroscopic scales*; e.g., turbulence, large molecules,... One might aim at (and sometimes succeeds in) *deriving* an effective theory from an underlying *microscopic theory* considered to be more fundamental ("Reductionism"). Most often, though, this exceeds our intellectual capacities.

Failure of Classical Physics & Birth of Modern Physics

It turned out that, without modifying one or both of these theories, it is impossible to unify Newtonian mechanics with Maxwell's theory of electromagnetism (Galilei- vs. Lorentz symmetry). Moreover, none of these theories allowed one to understand phenomena such as the *spectra of atoms*, e.g., *Balmer's* formula for the spectrum of a hydrogen atom. They did not enable one to understand the stability of atoms, molecules, condensed matter, stars (*Chandrasekhar limit*). They could not be used to describe magnetism, semi-conductors, superconductors, etc. On the basis of the theories of classical physics one cannot understand the photoelectric effect, the functioning of lasers, solar panels, ...

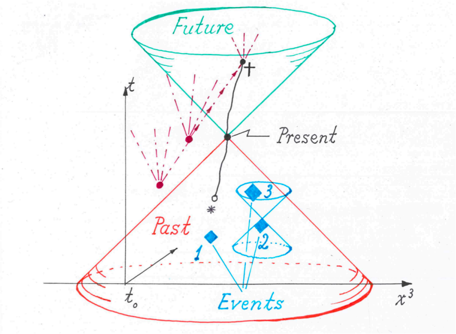
During the 20^{th} Century, these difficulties gave rise to two

Revolutions in theoretical physics:

1. The geometrical theories of *Special Relativity* (*Poincaré* and *Einstein*) unifying Newtonian mechanics and electromagnetism, and of *General Relativity* (*Einstein*) replacing Newton's law of universal gravitation by a more general, geometrical theory.

The Two Revolutions in Physics of the 20^{th} Century

One aspect of relativistic theories is that they are never fully predictive – for lack of complete knowledge of initial conditions, and hence $\exists$ a *fundamental dichotomoy* between past and future:



t_0: time right after inflation $\longrightarrow$ event horizon $\Rightarrow$ initial conditions not fully accessible!

Past = History of Actualities (Facts) / Future = Ensemble of Potentialities

This fundamental dichotomy should be and has been retained in:

2. *Quantum Mechanics* (the subject of the remaining lectures!)

Experimental Origins of Quantum Mechanics

Spectroscopy: Atoms and molecules only emit electromagnetic radiation of certain discrete frequencies.

Sichtbarer Bereich des Wasserstoff-Spektrums. Sichtbar sind sechs Linien der Balmer-Serie, da die CCD-Sensoren der Kamera auch ein wenig in den ultravioletten Teil des Spektrums hinein empfindlich sind.

Balmer-Rydberg Formel:

$$\frac{1}{\lambda} = R_\infty \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right), \qquad n_1 = 1, 2, \ldots, \; n_2 \geq n_1 + 1,$$

R_∞ = Rydberg constant; (Balmer found the formula for $n_1 = 2$.)

Blackbody radiation: Stefan-Boltzmann law; Wien's displacement law, Wien's formula for the spectral energy density of black-body radiation; Planck's law $\longrightarrow$ prediction of *photon* by Einstein $\longrightarrow$ his explanation of

Photoelectric Effect (Becquerel ... Hertz ... Millikan)

3. The "Dada" of Quantum Mechanics

"Paradoxically, [Dada's] *activities of deconstruction and destruction of languages translated itself into long-lasting works that opened up major new avenues ..."* (see *Larousse*. Dada: Cabaret Voltaire, Zurich 1916).

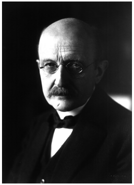
This reminds us of the role Quantum Mechanics has played in physics: It has deconstructed the language of Classical Physics and has opened up major new avenues towards understanding Nature.

Yet, the following amounts to an intellectual scandal:

"What we don't do is claim to understand Quantum Mechanics. Physicists don't understand their own theory any better than a typical smartphone user understands what's going on inside the device."

(Sean Carroll, in: *'New York Times'* 2019)

The *Heros* of Quantum Theory:



Planck

Einstein

Heisenberg

Dirac

The Beginnings of Quantum Theory

Quantum Theory started in 1900 with Planck's formula for the spectral energy density, $\rho(\nu, T)$, of black-body radiation, a discovery inspired by experimental data gathered in connection with work for the lighting of the streets of Berlin; i.e., it is an outgrowth of *applied science.*

$$\boxed{\rho(\nu, T) = \frac{8\pi}{c^3}\nu^2 \cdot \frac{h\nu}{e^{h\nu/k_B T} - 1}} \tag{3}$$

where ν = frequency, T = absolute temperature of radiation.
Fundamental constants: c: speed of light. h: Planck' constant, k_B: Boltzmann constant ($\propto \frac{1}{N_A}$); whence, with Newton's constant G_N,

$$\ell_{\mathbf{P}}{}^2 := \frac{G_N \cdot \hbar}{c^3} \qquad \text{(Planck length)}$$

The constants c, h, k_B, $\ell_{\mathbf{P}}$ stand for ≥ 4 *(past or future) Revolutions* in theoretical physics:

c: Special Relativity / h: Quantum Mechanics / k_B: Atomism & Statistical Mechanics / $\ell_{\mathbf{P}}$: General Relativity & Quantum Gravity (?)

Generalities -1: Physical Quantities in Quantum Mechanics

In all physical theories, **physical quantities** are represented by **"hermitian matrices"** (abstract self-adjoint operators).

In *Classical Physics*, the matrices representing physical quantities of a system S are given by real-valued functions on the state space, $\mathfrak{X}$, of S. They act as multiplication operators and generate an abelian algebra.

In *Quantum Mechanics (QM)*, physical quantities of S, such as its energy, momentum, angular momentum, particle number, etc. are represented by **non-commuting** hermitian matrices (*Heisenberg*, 1925)

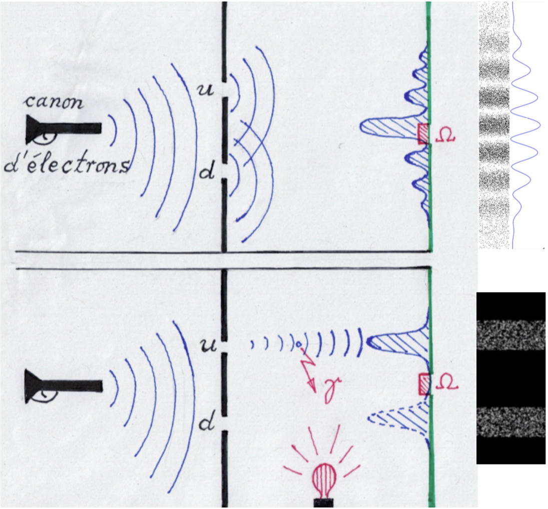
Thus, in *QM*, *physical quantities do usually* **not** *have precise values in any state of the system.* They have the meaning of *potentialities* (in the sense of *Aristotle*), and one can predict their values only with *certain probabilities*. Non-commuting pairs of physical quantities cannot be measured simultaneously, and their measured values obey the celebrated *Heisenberg uncertainty relations*.

Example: The color of a component of S (when illuminated by light) can be a physical quantitiy. In *QM*, such components are *chameleons:* Their colors are fundamentally *uncertain and depend on context!*

In *QM*, **potential events** are special physical quantities repesented by *families of orthogonal projections,* whose actions do **not** commute, either.

Experimental confirmation of these claims: Double-slit experiment – “interference”

In the left cavity, $\exists$ an electron gun, separated from right cavity by a double-slit screen; e^- hits the green screen with distr. as indicated:



$1 > prob\{e^- \text{ atteint } \Omega\} \gg 0$

$prob\{e^- \text{passe par } u \text{ et atteint } \Omega\} \approx 0$

$prob\{e^- \text{passe par } d \text{ et atteint } \Omega\} \approx 0$

Generalities - 2: Waves or particles? Waves **and** particles!

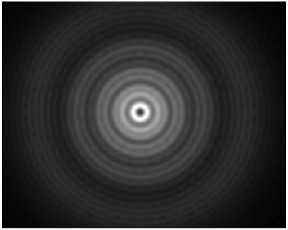
$$\Rightarrow \textit{prob}\{e^- \text{passes through } u \text{ \& arrives in } \Omega\} +$$

$$+\, \textit{prob}\{e^- \text{passes through } d \text{ \& arrives in } \Omega\} \overset{(*)}{\ll} \textit{prob}\{e^- \text{arrives in } \Omega\}$$

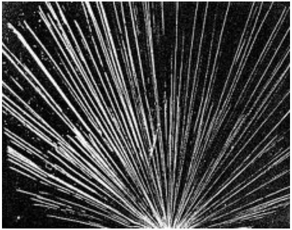
If projections representing potential events were commuting (e.g., in the presence of *"decoherence"*) we would observe *equality* in (*), because the sum of the projections repr. the events "e^- passes through u/d" is unity:

$$\text{"}e^- \text{ passes through } u + e^- \text{ passes through } d\text{"} = \text{unity} \qquad (!)$$

The *illumination* of the cavity to the right of the double slit by (e.g., laser) *light* has the effect that the *"electron wave"* (*de Broglie*) is converted to a *"corpuscle"* and the *quantum world* approaches the *classical world* (*Darwin, Mott, ...*; see also *Wheeler*'s "retarded choice")



Radioactive ball emitting α-particles, dark cavity



α-particle tracks, illuminated cavity

Generalities - 3: "The Problem of Hidden Variables in Quantum Mechanics"

"Die Logik nicht gleichzeitig entscheidbarer Aussagen" - E. Specker, 1960

La logique est d'abord une science naturelle. - F. Gonseth

"Kann die Beschreibung eines quantenmechanischen Systems durch Einführung von zusätzlichen – fiktiven – Aussagen so erweitert werden, dass im erweiterten Bereich die klassische Aussagenlogik gilt ... ? [meaning that all statements/results of experiments on the system could be embedded in a Boolean lattice.]

Die Antwort auf diese Frage ist **negativ***, ausser im Fall von Hilbertschen Räumen der Dimension 1 und 2. ... Ein elementargeometrisches Argument zeigt, dass eine solche Zuordnung* (such an embedding) **unmöglich** *ist, und dass daher über ein quanten-mechanisches System (von Ausnahmefällen abgesehen) keine konsistenten Prophezeiungen möglich sind."*

In his paper, Specker does not present any details concerning the "elementargeometrische Argument". They were provided in the famous paper by Kochen and Specker, seven years later, which I paraphrase next.

The Kochen-Specker Theorem

Simon Kochen and Ernst Specker, 1967

Question: $\exists$ a hidden-variables theory recovering the predictions of quantum mechanics; or, in other words, can the predictions of quantum mechanics be embedded in a Boolean lattice?

Let S be a physical system to be described quantum-mechanically. Its Hilbert space of pure state vectors is denoted by $\mathfrak{H}$; ...
If the answer to the above question were *"yes"* this would imply that $\exists$ a measure space $(\Omega, \mathfrak{F})$ and maps f and ρ,

$$f : A = A^* \in B(\mathfrak{H}) \mapsto f_A : \Omega \to \mathbb{R}, \ f_A \text{ is } \mathfrak{F}\text{-measurable} \tag{1}$$
$$\mu : \Psi \in \mathfrak{H} \mapsto \mu_{[\Psi]} = \text{ probability measure on } (\Omega, \mathfrak{F}),$$

with the following properties.

(P1) Preservation of expectation values: For every $A = A^* \in B(\mathfrak{H})$,

$$\|\Psi\|^{-2}\langle\Psi, A\Psi\rangle = \int_\Omega f_A(\omega)\, d\mu_{[\Psi]}(\omega)$$

Properties of a putative embedding in a Boolean lattice

(P2) If $u : \mathbb{R} \to \mathbb{R}$ is an arbitrary bounded measureable function then

$$f_{u(A)} = u \circ f_A.$$

Note: (P1) and (P2) are compatible with each other (spectral thm. – check!); and (P1) and (P2) imply the following fact:

(P3) Given any abelian algebra $\mathfrak{M}$ of commuting self-adjoint operators acting on $\mathfrak{H}$, then

$$f : A \in \mathfrak{M} \mapsto f_A \in L^\infty(\Omega)$$

is an *algebra homomorphism*; i.e.,

$$f_{A_1 \cdot A_2} = f_{A_1} \cdot f_{A_2}, \quad \forall A_1, A_2 \text{ in } \mathfrak{M}.$$

(Easy to prove if $\dim(\mathfrak{H}) < \infty$: all elements of $\mathfrak{M}$ are functions of a single matrix – check!)

The Kochen-Specker Theorem

As already noticed by Specker in 1960, a hidden-variables theory satisfying (P1) - (P3) exists if $\dim(\mathfrak{H}) = 1$ or 2, (QM of a spin-$\frac{1}{2}$ object – nowadays called "Qbit", which sounds more interesting).

Theorem. (Kochen & Specker, 1967)
If $\dim(\mathfrak{H}) \geq 3$, a hidden-variables theory satisfying (P1)-(P3) does **not** exist.

Proof. We consider a particle, whose spin degree of freedom is described by a vector operator, $\vec{S}$, acting on the Hilbert space $\mathfrak{H} = \mathbb{C}^3 \simeq \mathbb{R}^3 \otimes \mathbb{C}$, (i.e., the particle has spin 1). Let $(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ be the standard orthonormal basis in $\mathbb{R}^3$, and set $S_j := \vec{S} \cdot \vec{n}_j$, $j = 1, 2, 3$. Then

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \; S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One thus observes that the operators $P_j := 1 - S_j^2$, $j = 1, 2, 3$, are three mutually commuting orthogonal projections of rank 1, with $\sum_{j=1}^{3} P_j = 1$.

Arbitrary orthonormal bases in $\mathbb{R}^3$

More generally, for an arbitrary vector $\vec{e}$ in S^2, $P(\vec{e}) := 1 - (\vec{S} \cdot \vec{e})^2$ is an orthogonal projection projecting onto the one-dimensional subspace of $\mathfrak{H}$ spanned by $\vec{e}$. $\big[$Thus, the matrix elements of $P(\vec{e})$ in the basis $(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ are given by $P(\vec{e})_{ij} = e_i e_j\,,\ \forall i, j\,.\big]$

For an *arbitrary* orthonormal basis, $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, one then finds that

$$\sum_{j=1}^{3} P(\vec{e}_j) = 1\,, \quad P(\vec{e}_i) \cdot P(\vec{e}_j) = \delta_{ij} P(\vec{e}_i)\,, \tag{2}$$

but $[P(\vec{e}), P(\vec{e}\,')] \neq 0$, in general! The projections $\{P(\vec{e}_j)\}_{j=1}^{3}$ are functions of a single self-adjoint operator

$$A := \sum_{j=1}^{3} \alpha_j P(\vec{e}_j)\,, \quad \alpha_1 < \alpha_2 < \alpha_3\,. \tag{3}$$

generating a maximally abelian subalgebra of $B(\mathfrak{H}) = \mathbb{M}_3(\mathbb{C})$.

A fatal assumption

We now **assume** that ∃ a hidden-variables theory satisfying properties (P1), (P2) and (P3).

Since $P(\vec{e})^2 = P(\vec{e})$, it follows from (P2) that

$$P(\vec{e}) \mapsto f_{P(\vec{e})} =: \chi_{\vec{e}} \tag{4}$$

is a characteristic function on Ω. Eq. (2) implies that, for an arbitrary orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$,

$$\sum_{j=1}^{3} \chi_{\vec{e}_j} = 1, \quad \text{on } \Omega\,. \tag{5}$$

(For simplicity, we assume here and in the following that Ω is a discrete set.) For any point $\omega \in \Omega$,

$$\varphi_\omega(\vec{e}) := \chi_{\vec{e}}(\omega) \tag{6}$$

defines a function on S^2 with the following properties:

Strange functions on the unit sphere in $\mathbb{R}^3$

(i) It takes only the values 0 and 1, i.e.,

$$\varphi_\omega(\vec{e}) = 0 \text{ or } 1, \text{ for any unit vector } \vec{e} \in S^2.$$

(ii) If $\vec{e}$ belongs to any orthonormal basis $\left\{\vec{e}_1 \equiv \vec{e}, \vec{e}_2, \vec{e}_3\right\}$ of $\mathbb{R}^3$ then the value, $\varphi_\omega(\vec{e})$, of φ_ω on $\vec{e}$ should be **independent** of the choice of $\vec{e}_2$ and $\vec{e}_3$, and

$$\sum_{j=1}^{3} \varphi_\omega(\vec{e}_j) = 1\,.$$

This follows from Eqs. (5) and (6).

(iii) Properties (i) and (ii) imply that the function φ_ω is an *additive measure on the lattice of orthogonal projections* acting on $\mathbb{C}^3 = \mathbb{R}^3 \otimes \mathbb{C}$, $\forall\, \omega \in \Omega$.

Das "elementargeometrische Argument"

The evaluation of a function φ_ω with properties (i) - (iii) on finitely many unit vectors in $\mathbb{R}^3$, which give rise to finitely many orthonormal bases in $\mathbb{R}^3$, leads to the *contradiction* that, for some unit vectors $\vec{e}$, $\varphi_\omega(\vec{e}) = 0$ **and** $\varphi_\omega(\vec{e}) = 1$, depending on which completion of $\vec{e}$ to an orthonormal basis of $\mathbb{R}^3$ is considered – *"contextuality"*.

Kochen and Specker have found an explicit construction of finitely many unit vectors in S^2 leading to this contradiction. By now the best variant of their construction appears to require only 18 unit vectors.

There is an abstract proof of the claim that functions φ_ω on S^2 with properties (i) - (iii) do **not** exist, which is based on *Gleason's* theorem:[2]

Property (iii) says that the function φ_ω is an additive measure on the lattice of projections, $\forall\, \omega \in \Omega$. Gleason's theorem then says that

$\exists$ a *density matrix* $\Phi_\omega > 0$, with $\mathrm{tr}(\Phi_\omega) = 1$, such that

$$\varphi_\omega(\vec{e}) = \mathrm{tr}\big(\Phi_\omega\, P(\vec{e})\big) = \langle \vec{e}, \Phi_\omega\, \vec{e} \rangle\,.$$

This shows that $\exists$ a unit vector $\vec{e}$ such that $0 < \varphi_\omega(\vec{e}) < 1$. But this **contradicts** property (i)!

[2] I am grateful to *N. Straumann* for having explained this argument to me.

Connection to Kakutani's theorem

We note that Gleason's theorem apparently implies that the functions $\varphi_\omega(\vec{e})$ are **continuous** in $\vec{e}$.

Thus, let us consider an arbitrary real-valued, **continuous** function, φ, on the n-dimensional sphere, S^n, in $\mathbb{R}^{n+1}$ centered at the origin $\mathcal{O}$. *Dyson's* variant (Ann. Math. **54**, 534-536 (1951)) of *Kakutani's* theorem says that $\exists\, n+1$ points, $x_1, x_2, \ldots, x_{n+1}$, on S^n such that the $n+1$ unit vectors $\left\{\vec{e}_j := \overline{\mathcal{O}\, x_j} \mid j = 1, 2, \ldots, n+1\right\}$ are mutually orthogonal, and

$$\varphi(\vec{e}_1) = \varphi(\vec{e}_2) = \cdots = \varphi(\vec{e}_{n+1})$$

For $n = 2$, this contradicts properties (i) and (ii) of the functions φ_ω !

Remarks:

1. There is a variant of the Kochen-Specker theorem, due to *David Mermin*, based on studying 3 physical quantities (products of components of 3 spin-$\frac{1}{2}$ operators) with the following properties:

Gleason's theorem and Bell's inequalities

When the values measured for these quantities in a certain state of the system are multiplied, using props. (P1) – (P3), one obtains a number whose sign is **opposite** to the one of the number obtained when the multiplication is done using the rules of QM.

2. *Gleason's* theorem has been generalized by *Maeda*: Additive measures on the lattice of orthogonal projections of a *general* von Neumann algebra are given by *normal states* on the von Neumann algebra[3]. –

   Another, better known argument for the non-existence of hidden-variables theories of QM is based on:

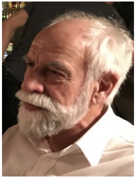
3. *Bell's inequalities*: These are inequalities on correlations between outcomes of some family of *commuting* measurements on two "causally independent" systems, A(lice) and B(ob). Bell's inequalities show that the range of quantum-mechanical correlations is *strictly larger* than the range of corresponding classical correlations. (I will explain this below.)

[3] see, e.g., *L. J. Bunce* & *J. D. Maitland Wright*, BAMS, **26**, 288-293 (1992)

Generalities - 4: The Phenomenon of Entanglement

Consider a composite system, $S = A \vee B$. *Alice*, A, can experiment on subsystem A, and *Bob*, B, can experiment on subsystem B. Their experiments "commute."

A



B

If A and B do their experiments *one after the other one,* they usually learn **strictly less** than if they do *simultaneous, coordinated* experiments!

A∨B

This phenomenon is a manifestation of *entanglement* and of what people call (somewhat misleadingly) the **"non-locality"** of *QM*.

A Manifestation of Entanglement: Bell's Inequalities

Let A and B be two (space-like separated) *"Qbits"*. A *maximally entangled* state of $A \vee B$ is given by the spin-singlet (or Bell) state

$$\Psi := \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right] \in \mathbb{C}^2_A \otimes \mathbb{C}^2_B\,. \qquad \text{(B1)}$$

We define a "correlation matrix," E, by setting

$$E(\vec{e}_1, \vec{e}_2) := \langle \Psi, A_{\vec{e}_1} \cdot B_{\vec{e}_2} \Psi \rangle,$$

where $A_{\vec{e}}$: spin component $\|\vec{e}$ in A, $B_{\vec{e}}$: spin component $\|\vec{e}$ in B.

In QM,

$$E(\vec{e}_1, \vec{e}_2) = -\,\vec{e}_1 \cdot \vec{e}_2\,. \qquad \text{(B2)}$$

We consider the following combination of correlations:

$$F(\vec{e}_1, \vec{e}_2, \vec{e}_1^{\,\prime}, \vec{e}_2^{\,\prime}) := E(\vec{e}_1, \vec{e}_2) + E(\vec{e}_1, \vec{e}_2^{\,\prime}) + E(\vec{e}_1^{\,\prime}, \vec{e}_2) - E(\vec{e}_1^{\,\prime}, \vec{e}_2^{\,\prime}). \qquad \text{(B3)}$$

Existence of *"local" hidden variables* in QM would imply that

$$-2 \leq F(\vec{e}_1, \vec{e}_2, \vec{e}_1^{\,\prime}, \vec{e}_2^{\,\prime}) \leq 2\,, \qquad \text{(B4)}$$

$\forall$ choices of $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_1^{\,\prime}$ and $\vec{e}_2^{\,\prime}$; (an easy exercise!). But, in QM, one can choose $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_1^{\,\prime}$ and $\vec{e}_2^{\,\prime}$ such that

$$\boxed{F(\vec{e}_1, \vec{e}_2, \vec{e}_1^{\,\prime}, \vec{e}_2^{\,\prime}) = 2\sqrt{2}\ !} \qquad \text{(B5)}$$

Proof of Bell's Inequalities

To show (B2), choose $\vec{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} \sin\theta \sin\varphi \\ \sin\theta \cos\varphi \\ \cos\theta \end{pmatrix}$, with $\varphi = 0$.

If (classical) local hidden variables existed then

$$E(\vec{e}_1, \vec{e}_2) = \int_\Omega a_{\vec{e}_1}(\omega) \cdot b_{\vec{e}_2}(\omega) d\mu_{[\Psi]}(\omega), \qquad \text{(B6)}$$

for some probability measure $d\mu_{[\Psi]}$; (I find (B6) an absurd assumption!)
Since $A^2_{\vec{e}_1} = \mathbf{1}$ and $B^2_{\vec{e}_2} = \mathbf{1}$, we have that

$$|a_{\vec{e}_1}(\omega)| = 1 \quad \text{and} \quad |b_{\vec{e}_2}(\omega)| = 1, \quad a.\ e.$$

Recalling definition (B3) of $F(\vec{e}_1, \vec{e}_2, \vec{e}_1', \vec{e}_2')$, applying (B6) to each term in $F(\vec{e}_1, \vec{e}_2, \vec{e}_1', \vec{e}_2')$, and using the elementary inequality (exercise)

$$-2 \leq x\,y + x\,y' + x'\,y - x'\,y' \leq 2, \qquad \text{(B7)}$$

for arbitrary x, y, x' and y' in the interval $[-1, 1]$, we conclude that (B4) holds.

Proof completed

[Note that

$$x\,y + x\,y' + x'\,y - x'\,y' = x(y + y') + x'(y - y'),$$

and, under our hypotheses, $|x| \leq 1$, $|y + y'| + |y - y'| \leq 2$. Hence (B7) follows.]

Choosing, e.g.,

$$\vec{e}_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{e}_2 := \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\vec{e}_1^{\,\prime} := \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{e}_2^{\,\prime} := \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \tag{B8}$$

and using definition (B3) of F and identity (B2) with (B8), we find that, *according to QM*,

$$F(\vec{e}_1, \vec{e}_2, \vec{e}_1', \vec{e}_2') = 4 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2} \quad \rightarrow \quad (B4) \text{ violated!}$$

More general results on Bell's inequalities

$S = S_1 \vee S_2$, with $\mathcal{H}_S = \mathcal{H}_1 \otimes \mathcal{H}_2$. Families of operators

$$\mathcal{D}_i := \{ O \in B(\mathcal{H}_i) \mid O^* = O, \|O\| \leq 1 \}, \quad i = 1, 2.$$

If *local hidden variables* existed then $\mathcal{D}_i \;\to\; \mathcal{D}_i(\Omega, \mu)$ (real random variables on Ω bounded in absolute value by 1), with

$$\begin{aligned} \mathcal{D}_1 \ni A \;\mapsto\; a \in \mathcal{D}_1(\Omega, \mu), \\ \mathcal{D}_2 \ni B \;\mapsto\; b \in \mathcal{D}_2(\Omega, \mu). \end{aligned} \tag{B9}$$

Correlation matrices:

$$\begin{aligned} \mathcal{M}_Q^{KL} &:= \{ \Gamma \mid \Gamma_{k\ell} := \operatorname{tr}(\rho\, A_k \otimes B_\ell), k = 1, \ldots, K, \ell = 1, \ldots, L \} \\ \mathcal{M}_C^{KL} &:= \{ \gamma \mid \gamma_{k\ell} := \int_\Omega a_k(\omega) b_\ell(\omega) d\mu(\omega), k = 1, \ldots, K, \ell = 1, \ldots, L \}, \end{aligned} \tag{B10}$$

where $A_k \in \mathcal{D}_1$, $a_k \in \mathcal{D}_1(\Omega, \mu)$, $k = 1, \ldots, K$, $B_\ell \in \mathcal{D}_1$, $b_\ell \in \mathcal{D}_2(\Omega, \mu)$, $\ell = 1, \ldots, L$; ρ is a density matrix on $\mathcal{H}_S$, $d\mu$ a probability meas. on Ω.

Tsirelson's Theorem

Theorem: *Let $\Gamma \in \mathcal{M}_Q^{KL}$. Then there is a constant, $K_G > 1$, such that*

$$\gamma := K_G^{-1}\,\Gamma \in \mathcal{M}_C^{KL}\,, \text{ for arbitrary } K, L. \qquad \square$$

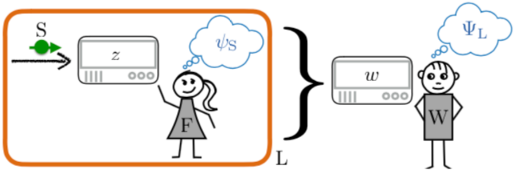
The constant K_G (> 1) has been introduced and estimated by Grothendieck in his work on tensor products of topological vector spaces and is therefore called *"Grothendieck constant"*. The exact value of K_G is not known. A known upper bound due to Krivine (which has been shown not to be strict) is

$$K_G < \frac{\pi}{2\log(1+\sqrt{2})} \approx 1.782.$$

As an exercise you may try to choose spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, operators $A_1, \ldots, A_K, B_1, \ldots, B_L$ and corresponding classical random variables $a_1, \ldots, a_K, b_1, \ldots, b_L$ such that $\gamma \neq \Gamma$; (see, e.g., Mermin's work on Bell's inequalities!)

Generalities - 4: The Schrödinger Eq. does **Not** Describe the *Time Evolution* of *States of Individual Systems* in QM

(I) To understand this, we consider, for example, the *Wigner's friend paradox* (see Wigner; Hardy; Frauchiger-Renner; ...):



Courtesy *Frauchiger & Renner*

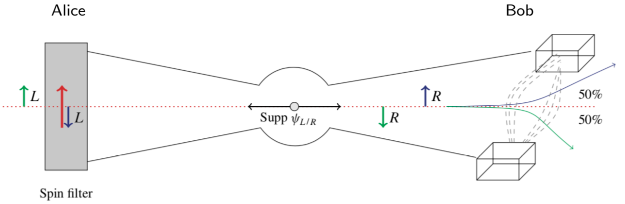
F measures the spin of the green particle in the *vertical* direction. After a **successful** measurement, states of F and of the particle are entangled. F makes predictions about future measnts. using a mixed state, while W uses **unitary evolution** of **pure** initial state of entire lab, including F, to make his predictions. Then the *statistics of future measurement outcomes* predicted by F and W are **contradictory**. – Well, this only shows that: The state of the lab evolves **non-linearly**; W's predictions are *wrong*.

Quantum theory cannot be fully predictive, because ...

(II) A Gedanken-Experiment (due to Faupin-F-Schubnel) that is, perhaps, more compelling than *"Wigner's friend,"* or the F-R version thereof:

Two particles (silver atom & electron), P and P', prepared in spin-singlet initial state, $\psi_{L/R}$,[4] with orbital wave functions chosen such that P propagates into the cone opening to the right, while P' propagates into the cone opening to the left and ending in a detector behind a spin filter, (except for very tiny tails leaking beyond these cones).

As a consequence of *cluster properties* of propagator the **time evolution** of P is ess. *independent* of the one of $Q := P' \vee$ *spin filter* $\vee$ *detector*



ess. support of orbital wave fu. of P' | ess. support of orbital wave fu. of P

[4] nowadays called a *Bell pair of Qbits*

... because Quantum theory is fundamentally probabilistic

Temporary assumptions (leading to a contradiction!):

i. P and P': Spin-$\frac{1}{2}$ particles prepared in a *spin-singlet initial state*; spin filter prepared in a poorly known initial state not (necessarily) entangled with initial state of P' and P.

ii. *Dynamics of state of total system fully determined by Schrödinger equation.* In particular, initial state of spin filter determines whether P' will pass through it or not, (given that the initial state of $P' \vee P$ is a spin-singlet state, with P' and P moving into opposite cones).

iii. *Correlations between outcomes of spin measurements of P' and of P are as predicted by standard quantum mechanics*, (invoking, e.g., "Copenhagen interpretation") – "non-locality" of QM.

Fact: For short-range interactions, the Schrödinger evolution of the state of the system factorizes into ess. free evolution of P, tensored with the evolution of $Q := \{P' \vee$ spin filter $\vee$ detector$\}$, up to tiny errors. This follows from choice of initial conditions & cluster props. of propagator! Hence the *spin of P* is ess. *conserved* before its measurement!

… and, yes, the Schrödinger Equation does *not* describe the time evolution of *states* of individual systems in QM!

$\Rightarrow$ If the evolution of the state of the *total* system were fully determined by a Schrödinger equation then:
Expectation value of spin of $P \approx 0, \forall$ times! $\Rightarrow$ The state of the spin of *P' after interaction of P' with spin filter & detector* does **not** bias the state of the spin of P when measured, (e.g., in a Stern-Gerlach exp.)!

Thus, if the usual *correlations* between two "independent" measurements (here of z-comp. of spins of P' and of P), predicted on the basis of the projection postulate of "Copenhagen", are observed[5] then it follows that the *Schrödinger equation, and nothing more than it, cannot* describe the evolution of states of individual systems when measurements occur.

$\rightarrow$ We must find the correct probabilistic law of evolution of states in *QM* that replaces Schrödinger evolution !

However, it is safe to assume the validity of *Heisenberg-picture evolution* of operators representing physical quantities of *isolated systems* (define!), which is perfectly *deterministic* (while the *evolution of states* is stochastic) . $\Rightarrow$ Equivalence of the *Heisenberg picture* and the standard *Schrödinger picture* is an **erroneous claim**!

[5] as suggested by the experiments of *Aspect* and others

Generalities - 5: "Non-locality" of QM versus "Einstein causality"

It is possible that the measurements of components of the spins of P and of P' are made in *space-like separated regions* of space-time, so that the localization regions of the corresp. projection operators, $\Pi^{P'}_{\sigma',\vec{e}_z}$ and $\Pi^{P}_{\sigma,\vec{n}}$, with $\sigma = \pm, \sigma' = \pm$, are space-like separated. The order in which these two measurements occur then depends on the rest frame of the observer who records the measurement data. This implies that the operators $\Pi^{P'}_{\sigma',\vec{e}_z} \cdot \Pi^{P}_{\sigma,\vec{n}}$ and $\Pi^{P}_{\sigma,\vec{n}} \cdot \Pi^{P'}_{\sigma',\vec{e}_z}$ must have the *same effect* when applied on the state of the system. ... The most general way in which this can be guaranteed is to require that

$$\boxed{\Pi^{P'}_{\sigma',\vec{e}_z} \cdot \Pi^{P}_{\sigma,\vec{n}} = \Pi^{P}_{\sigma,\vec{n}} \cdot \Pi^{P'}_{\sigma',\vec{e}_z}} \tag{1}$$

This is *locality*, or *Einstein causality*, of quantum theory ($\nearrow$ RQFT). It fits perfectly into QM!

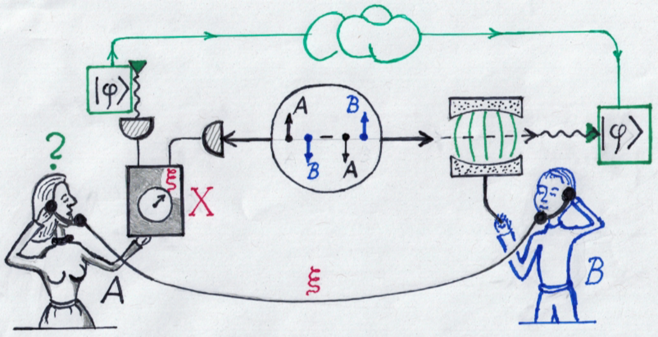
—

The *"non-locality of QM"* is often misrepresented, and the talk about a tension between QM and Relativity Theory is misguided and misleading.

Quantum Teleportation – *Bennet, Brassard* et al.

This is a quantum phenomenon **impossible** in the classical world.

In a central region, O, one prepares a spin-singlet state of two particles of spin $\frac{1}{2}$. One of them, labelled "A," is sent to *Alice*, the other one, labelled "B," is sent to *Bob* (as in the previous thought experiment).



Simultaneously, *Alice* catches a second particle of spin $\frac{1}{2}$ in a state $|\varphi\rangle \equiv |\underset{A}{\varphi}\rangle$ unknown to her. She then measures the value of a *physical quantity*, X, in the composite state $|\underset{A}{\varphi}\rangle \otimes |\underset{A}{\bullet}\rangle$, of the two particles:

Quantum Teleportation proves the *"non-locality"* of *QM*

Let ξ be the value of X measured by *Alice*. She calls Bob by phone to communicate to him this value, an information that is transmitted purely classically. Knowledge of ξ motivates *Bob* to perform a ξ-dependent operation (precession) on the spin of the particle he had received, whose state was actually already affected by *Alice*'s measurement of X – **"non-locality"** of *QM*!

$\Rightarrow$ Effect of measurement of X by Alice, with value ξ, and of *Bob*'s ξ-dependent operation on the state of the particle he had received yields the following transformation of the state of *Bob*'s particle:

$$|\underset{B}{\bullet}\rangle \overset{\text{measnt. of } X}{\longrightarrow} |\underset{B}{\text{intermediate state, dep. on } \xi}\rangle \overset{\xi\text{-dep. precession}}{\longrightarrow} |\underset{B}{\varphi}\rangle$$

<u>*Conclusion:*</u> The state $|\varphi\rangle$ of the second particle received by *Alice*, *unknown to her*, has been teleported to the particle received by *Bob*!

The mathematical details met in the analysis of teleportation represent an elementary (but somewhat clever) exercise in linear algebra.

Some mathematical details on quantum teleportation

Our arguments rely on the Copenhagen interpretation of QM: Let Φ be the initial state of a system S. If an "observable," $X = \sum \xi \Pi_{\xi}$, of S is measured with outcome $\xi_* \in \sigma(X)$ then the state of S immediately after the measurement of X is given by

$$\Pi_{\xi_*}\Phi / \|\Pi_{\xi_*}\Phi\| , \tag{T1}$$

(*Lüders'* projection postulate). The probability to measure ξ_* in state Φ is given by

$$prob\{\xi_* | \Phi\} = \|\Pi_{\xi_*}\Phi\|^2 \qquad \text{(Born's Rule)} \tag{T2}$$

—

Alice and Bob capture, each, one particle of a Bell pair initially prepared in the state Ψ introduced in Eq. (4) above, denoted A and B, respectively. Alice also captures a second particle, C, with spin $\frac{1}{2}$ whose spin is in some state

$$\varphi := \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^2, \quad |u|^2 + |v|^2 = 1,$$

unknown to Alice. We define $S := (C \vee A) \vee B$.

Further details

The initial state of S is given by

$$\Phi := \varphi \otimes \Psi = \frac{1}{\sqrt{2}} \Big\{ \varphi \otimes |\uparrow\rangle_A \otimes |\downarrow\rangle_B - \varphi \otimes |\downarrow\rangle_A \otimes |\uparrow\rangle_B \Big\} \qquad \text{(T3)}$$

Alice now measures an "observable" X given by

$$X = \Pi_1 + 2\Pi_2 + 3\Pi_3 + 4\Pi_4, \quad \text{with} \quad \Pi_i := |\chi_i\rangle\langle\chi_i|\,, \qquad \text{(T4)}$$

of the sub-system $A \vee C$, where

$$\begin{aligned}
\chi_1 &:= \frac{1}{\sqrt{2}} \Big\{ |\uparrow\rangle_C \otimes |\downarrow\rangle_A - |\downarrow\rangle_C \otimes |\uparrow\rangle_A \Big\} \quad (s = 0) \\
\chi_2 &:= \frac{1}{\sqrt{2}} \Big\{ |\uparrow\rangle_C \otimes |\downarrow\rangle_A + |\downarrow\rangle_C \otimes |\uparrow\rangle_A \Big\} \quad (s = 1,\ m = 0) \\
\chi_3 &:= \frac{1}{\sqrt{2}} \Big\{ |\uparrow\rangle_C \otimes |\uparrow\rangle_A - |\downarrow\rangle_C \otimes |\downarrow\rangle_A \Big\} \quad (s = 1) \\
\chi_4 &:= \frac{1}{\sqrt{2}} \Big\{ |\uparrow\rangle_C \otimes |\uparrow\rangle_A + |\downarrow\rangle_C \otimes |\downarrow\rangle_A \Big\} \quad (s = 1) \qquad \text{(T5)}
\end{aligned}$$

[Note that $\frac{1}{\sqrt{2}}(\chi_3 + \chi_4) = |\uparrow\rangle_C \otimes |\uparrow\rangle_A$, i.e., $s = 1, m = 1$, etc.]

More details

When Alice has measured the "observable" X the state of $S = (C \vee A) \vee B$ is given by one of the following rays (normalization unimportant):

$$\begin{aligned}
[\Pi_1 \varphi \otimes \Psi] =& \chi_1 \otimes \Big\{ -\langle \uparrow | \varphi \rangle_C | \uparrow \rangle_B - \langle \downarrow | \varphi \rangle_C | \downarrow \rangle_B \Big\} \\
[\Pi_2 \varphi \otimes \Psi] =& \chi_2 \otimes \Big\{ -\langle \uparrow | \varphi \rangle_C | \uparrow \rangle_B + \langle \downarrow | \varphi \rangle_C | \downarrow \rangle_B \Big\} \\
[\Pi_3 \varphi \otimes \Psi] =& \chi_3 \otimes \Big\{ +\langle \uparrow | \varphi \rangle_C | \downarrow \rangle_B + \langle \downarrow | \varphi \rangle_C | \uparrow \rangle_B \Big\} \\
[\Pi_4 \varphi \otimes \Psi] =& \chi_4 \otimes \Big\{ +\langle \uparrow | \varphi \rangle_C | \downarrow \rangle_B - \langle \downarrow | \varphi \rangle_C | \uparrow \rangle_B \Big\}
\end{aligned} \tag{T6}$$

Each measurement outcome, $i = 1, 2, 3, 4$, has the same a-priori probability given by $\|\Pi_i \varphi \otimes \Psi\|^2 = \frac{1}{4}$.
Alice communicates the measurement result, ξ_*, **classically** (e.g., by telephone) to Bob. If she has measured the value 1 for X, i.e., S is in state $[\Pi_1 \varphi \otimes \Psi]$, then, according to the first equation above, particle B in Bob's lab is in state $[-\varphi] = [\varphi]$, and Bob doesn't take any further action.

If Alice measures 2 and communicates this result to Bob then ...

Yet some further details

... Bob performs a spin precession by an angle of 180° around the 3-axis on particle B. This maps the states $[\Pi_2 \varphi \otimes \Psi]$ to

$$\left[\left(\mathbf{1}_{C \vee A} \otimes i\sigma^3_B\right) \cdot \Pi_2 \varphi \otimes \Psi\right] = \left[\chi_2 \otimes i\left\{ -\langle \uparrow | \varphi \rangle_B - \langle \downarrow | \varphi \rangle_C | \downarrow \rangle_B \right\}\right]$$

Thus, particle B in Bob's lab is now in state $[-i\varphi] = [\varphi]$.
If Alice measures the value 3 for X then Bob performs a spin precession by an angle of 180° around the 1-axis on particle B, which then ends up in state $[i\varphi] = [\varphi]$.
Finally, if Bob learns from Alice that she has measured the value 4 for X he performs a spin precession by an angle of 180° around the 2-axis on particle B, and B then ends up in state $[\varphi]$.
In all four instances, particle B in Bob's lab ends up in state $[\varphi]$; i.e., the initial state, $[\varphi]$, of particle C captured by Alice (and unknown to her) has been teleported to particle B in Bob's lab.
Note, however, that, for teleportation to work, Alice has to communicate (classically) to Bob the value she has measured for the "observable" X!
(Generalizations to higher-dim. Hilbert spaces are straightforward.)

Experimental realization of teleportation: *Zeilinger* et al., 1997.

4. Time, Events and States in *QM*

So far, everything has been fairly clear. For, we have not spoken about the role of **time** in *QM* and about the infamous *measurement problem*, yet. Let's conclude this lecture with a few comments on these crucial matters! – Henceforth I focus attention on *isolated* physical systems...

The three conventional pillars QM rests upon:

1. In *QM*, *"potential events"* that might (but need not) happen in the future are descriibed by a *classical alternative*, namely by a partition of unity, $\mathfrak{F}$, by disjoint orthogonal projections, π, acting on some Hilbert space whose sum equals unity, $\mathbf{1}$.
2. In the *Heisenberg picture*, the dependence of *potential events* on the *time* of their possible occurrence is described by the well known *Heisenberg equations*, which are perfectly **deterministic**. – *How then does* **randomness** *enter QM?*
3. In *QM*, the **"state"** of a physical system at time t is described by a *"quantum probability measures"*, ω_t, that assigns to every projection π representing a potential event that might occur at time t, or later, an **a-priori probability of occurrence** denoted by

$$\omega_t(\pi) \in [0, 1], \qquad \text{(Born's Rule)}$$

The Fourth Pillar of *QM*: Decline of Potentialities

"Indeed, it is evident that the mere passage of time itself is destructive rather than generative [...], *because change is primarily a 'passing away'."* (Aristotle, Physics)

Our task is now to formulate a precise *Law* that determines whether a potential event possibly occurring at time t, in the sense described above, has a chance to **actually occur** at a time $\geq t$.

Let $\mathcal{E}_{\geq t}$ consist of *all operators* arising (by taking linear combinations of products) from projections, π, representing potential events possibly occurring at time t or later. One can argue convincingly that, because of interactions of electrically charged matter with the quantized electromagnetic field, the following

4. *"Principle of Declining Potentialities" (PDP)*

$$\boxed{\mathcal{E}_{\geq t'} \subsetneqq \mathcal{E}_{\geq t}, \quad \forall\, t' > t}$$

holds. Thanks to this principle and to the phenomenon of *entanglement*, a *state*, ω_t, determines a *unique* partition of unity $\mathfrak{F}(\omega_t) \subset \mathcal{E}_{\geq t}$ by *potential events* ... of which **one** will actually happen (actualize) at time t:

State Reduction Postulate & Evolution of States

State Reduction Postulate: At every time t, some event $\pi_{*,t} \in \mathfrak{F}(\omega_t)$ **actually happens**. *The* **probabilty**, *prob($\pi_{*,t}$), that $\pi_{*,t}$ happens, predicted by Quantum Mechanics, is given by* Born's Rule *(BR):*

$$\boxed{\text{prob}(\pi_{*,t}) = \omega_t(\pi_{*,t}), \qquad \pi_{*,t} \in \mathfrak{F}(\omega_t)}$$

Let $dt > 0$ be the *"time step."* The state ω_t and the event $\pi_{*,t}$ uniquely determine a state ω_{t+dt} at time $t + dt$ in the range of $\pi_{*,t}$, which then determines a unique partition of unity $\mathfrak{F}(\omega_{t+dt}) \subset \mathcal{E}_{\geq t+dt}$ by potential events, of which one, $\pi_{*,t+dt}$, actually happens at time $t + dt$; etc.

Thus, in the *Heisenberg picture*, the *evolution of states* of an isolated physical system follows a stochastic *history (H)* of random *events (E)*, $\{\pi_{*,t}\}_{t\in\mathbb{R}}$, on a *tree-like structure (T)*:

$(\omega_t, \pi_{*,t}) \to$ state $\omega_{t+dt} \overset{PDP}{\to}$ partition $\mathfrak{F}(\omega_{t+dt})$ of $\mathbf{1}$ by possible events

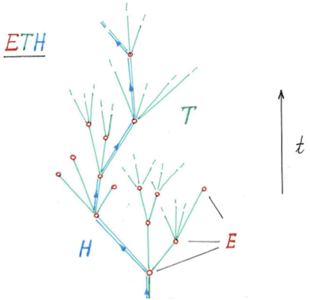
$\overset{SRP}{\to}$ an actual event, $\pi_{*,t+dt} \in \mathfrak{F}(\omega_{t+dt})$, happens (with Born Rule!)

$$(\omega_{t+dt}, \pi_{*,t+dt}) \to \omega_{t+2dt} \overset{PDP}{\to} \text{partition } \mathfrak{F}(\omega_{t+2dt}) \text{ of } \mathbf{1} \quad \cdots$$

This has motivated me to call this formalism "*ETH* Approach to *QM*":

A Metaphorical Picture of "*ETH*"

The "*ETH*-Approach is the subject of further lectures.



Have I lost some of you in the subtleties of the *"dada"* of *QM*?

Let us not forget what Galileo has taught us: *"The Book of Nature is written in mathematical language."*