Quantum Poisson Processes[1]

Das andere große Problem, mit dem ich mich seit etwa 1900 befasst habe, ist das der Strahlungs- und Quantentheorie. . . . Den Rest meines Lebens werde ich wohl der grundsätzlichen Klärung dieses Problems widmen, wie gering auch die Aussichten auf ein Erreichen dieses Zieles erscheinen mögen. (Albert Einstein, 1924)

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I thank the organizers for the opportunity to present this material ...

Summary

The constant element in physics, since Newton, is not a configuration or a geometrical form, but a **law of dynamics**.
(Werner Heisenberg)

This is an attempt to find a *Law of Dynamics* for non-relativistic Quantum Mechanics.

The time evolution of physical systems consisting of charged matter interacting with the quantized electromagnetic field, e.g., electrons in the shell of an atom coupled to the radiation field, is **dissipative** and **stochastic**, featuring isolated random events sometimes called **"quantum jumps"**.

In an idealized description of the time evolution of such systems a new type of stochastic process is encountered, which I call *"quantum Poisson process"*. In this talk, I introduce this process, present some of its mathematical properties and sketch some applications to quantum theory.

1. Diffusion and the theory of random walkers

In order to set the stage for the main topic of my talk – namely the theory of *quantum Poisson processes* with applications to Quantum Mechanics – we consider the **theory of random walkers** on a lattice $\mathbb{Z}^\nu, \nu = 1, 2, 3, \ldots$, parametrized by time τ.

Ontology: At every time $\tau \in [0, t), t > 0$, with the exception of finitely many randomly chosen times $t_1, \ldots, t_n$, with $0 < t_1 < t_2 < \cdots < t_n < t$, $n = 0, 1, \ldots$, a random walker on $\mathbb{Z}^\nu$ occupies a unique, though randomly chosen site given by $x_\omega(\tau) = \omega(k) \in \mathbb{Z}^\nu$, for $t_k < \tau < t_{k+1}$, with $k = 0, 1, \ldots, n$, $(t_0 = 0, t_{n+1} = t)$, $n =: \ell(\omega)$ is the length of a simple random walk ω. At time $\tau = t_k$, $x_\omega(\tau)$ jumps from $\omega(k-1)$ to $\omega(k)$, which is a *nearest neighbor* of $\omega(k-1)$ (i.e., $|\omega(k) - \omega(k-1)| = 1$.)

Law: The probability that a random walker at $x_\omega(\tau) = \omega(k)$ does **not** jump to a neighboring site $\omega(k+1)$, for $\tau \in [s, s + ds)$, is given by

$$\begin{aligned} & p_{nj}[s, s + ds] = 1 - 2\nu \cdot ds \\ \Rightarrow \quad & p_{nj}[t_1, t_2] = e^{-2\nu(t_2 - t_1)}, \text{ for } t_1 < t_2, \quad nj : \text{“no jumps”}. \end{aligned} \tag{1}$$

A Poisson jump process

Let $\delta \in \{\pm\vec{e}_1, \ldots, \pm\vec{e}_\nu\}$ be a direction in the lattice. The probability for making a jump in direction δ in the time interval $|s, s+ds)$ is given by

$$p_\delta[s, s+ds] = ds. \tag{2}$$

Hence

$$p_{nj}[s, s+ds] + \sum_{\delta \in \{\pm\vec{e}_1, \ldots, \pm\vec{e}_\nu\}} p_\delta[s, s+ds] = 1\,.$$

Suppose $x_\omega(\tau)$ is a random walker starting at $\tau = 0$ at an arbitrary site $\omega(0) \in \mathbb{Z}^\nu$ and making $n = \ell(\omega)$ jumps along a given SRW ω, at times $\tau \in [t_k, t_k + dt_k)$, $k = 1, \ldots, n$, until it stops at time t. The probability of encountering such a random walker is then given by

$$\begin{aligned} W_\omega\left[t_1, \ldots, t_n\right] \prod_{j=1}^{n} dt_j :=& \\ =& \Big\{ \prod_{k=1}^{n} p_{nj}[t_{k-1}, t_k]\, p_{\omega(k)-\omega(k-1)}[t_k, t_k + dt_k] \Big\} p_{nj}[t_n, t] \\ =& e^{-2\nu\cdot t} dt_1 \ldots dt_n\,, \quad \text{where} \quad t_0 = 0 < t_1 < \cdots < t_n < t\,. \end{aligned} \tag{3}$$

Diffusive motion

Consequences of the Law:

(i) Averaging over all possible SRW's, ω, beginning at a site x and ending at a site y, one finds that

$$\sum_n \sum_{\omega: x \to y,\, \ell(\omega)=n} \underbrace{e^{-2\nu\cdot t} \int_{\Delta_n[0,t]} dt_1 \dots dt_n}_{=\frac{t^n}{n!} e^{-2\nu\cdot t}} =$$

$$= \left(e^{t\Delta}\right)_{xy} \quad \to \text{heat kernel!}$$

(ii) Furthermore,

$$\mathbb{E}_\omega \left[x_\omega(t) - x_\omega(0)\right]^2$$

$$= \sum_n \underbrace{\mathbb{E}_\omega \left[\omega(n) - \omega(0)\right]^2}_{=n} \frac{(2\nu\cdot t)^n}{n!} e^{-2\nu\cdot t}$$

$$= 2\nu\cdot t = \sum_{y\in\mathbb{Z}^\nu} \left(e^{t\Delta}\right)_{xy} \left[y-x\right]^2.$$

$\to$ random walkers perform *diffusive motion!* Etc.

Diffusion equation and increase of entropy

Let $\rho_t(x)$ be the density of random walkers at time t, i.e, an "ensemble average" over the states of random walkers at time $\tau = t$). By (i),

$$\dot{\rho}_t(x) = \big(\Delta\rho_t\big)(x) \quad \Leftrightarrow$$

$$\rho_{t+dt}(x) = \rho_t(x) + \sum_{\delta\in\{\pm\vec{e}_1,\ldots,\pm\vec{e}_\nu\}} \big[\rho_t(x+\delta) - \rho_t(x)\big]\,dt + \mathcal{O}(dt^2) \qquad (4)$$

Eq. (4) implies that $\sum \rho_t(x) = \sum \rho_0(x)$ is *independent* of t, as expected. By **"unraveling"** Eq. (4) we recover the Law (1) - (3) of random walkers.

We define the *entropy* of a density ρ on $\mathbb{Z}^\nu$, with $\sum \rho(x) = 1$, by

$$S[\rho] := -k \sum_{x\in\mathbb{Z}^\nu} \rho(x)\,\ell n(\rho(x))\,.$$

Theorem: *If ρ_t satisfies the diffusion equation (4), with $\sum_x \rho_t(x) = 1$, then $S[\rho_t]$ is a monotone* **increasing** *function of time t.*

Proof is an exercise (using that the heat kernel is positivity preserving, hence $\rho_0(x) \geq 0 \Rightarrow \rho_t(x) \geq 0, \forall x$, as well as "summation by parts").

Interpretation: Motion of random walkers is a **disspative** process, i.e., in average, it produces entropy.

2. Systems of matter interacting with the quantized radiation field

To be specific, we consider a physical system, S, consisting of a *static atom* (i.e., orbital motion neglected) with $N = 2, 3, \ldots$ internal states of energies $E_0 < \cdots < E_{N-1}$, which are described by a complete orthonormal system, $\{\psi_0, \ldots, \psi_{N-1}\}$, of eigenstates of a matrix H_A, the *Hamiltonian* of the atom, acting on the Hilbert space $\mathfrak{H}_A := \mathbb{C}^N$:

$$H_A \psi_j = E_j \psi_j, \qquad \text{for } \; j = 0, 1, \ldots, N-1 \,.$$

One may suppose the atom is coupled to the quantized electromagnetic field and specify *transition amplitudes,* d_{ij}, between the state vectors ψ_i and ψ_j, $i, j = 0, 1, \ldots, N-1$, with $d_{ij} = \overline{d_{ji}} \in \mathbb{C}$. The transition from ψ_i to ψ_j is accompanied by *emission* of a photon of frequency $\omega_{ij} \approx \hbar^{-1}[E_i - E_j]$ if $i > j$, and by *absorption* of a photon of frequency $\omega_{ij} \approx \hbar^{-1}[E_j - E_i]$ if $i < j$. State space of photons = Fock space, $\mathcal{F}$.

We will describe the radiation field in a limiting regime corresponding to $c \to \infty$, where c is the velocity of light, and we assume for simplicty that there are no detectors in S recording the passage/emission of photons.

Dynamics of atom coupled to the radiation field

In text books, the dynamics of the atom coupled to the quantized radiation field is described by unitary evolution on the Hilbert space $\mathfrak{H} := \mathfrak{H}_A \otimes \mathcal{F}$ generated by a *Hamiltonian*

$$H := H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_f + eH_I(\{d_{ij}\}) \,,$$

where H_f is the Hamiltonian of the non-interacting electromagnetic field, e is the elementary electric charge, and H_I is an operator on $\mathfrak{H}$ describing the transitions between internal states of the atom accompanied by the emission or absorption of photons, as alluded to above.

If $c < \infty$ and $\tau > 0$ we introduce algebras (see last lecture, Eq. (18))

$$\mathcal{E}_{\geq t} \simeq B(\mathfrak{H}_A) \otimes \mathfrak{D}_{\geq t} \,, \tag{5}$$

where $\mathfrak{D}_{\geq t}$ is the algebra generated by all bounded functions of electromagnetic field operators localized in the future of the space-time point $(\mathbf{0}, t)$. A *state* of the system S *at time* t is given by a state on the algebra $\mathcal{E}_{\geq t}$.

Eliminating the degrees of freedom of the radiation field from our description

In the limit where the velocity of light $c \to \infty$, we may let $\tau \searrow 0$, the algebras $\mathcal{E}_{\geq t}$ are type-I_∞, and *PDP* takes the following form:

$$\left[\mathcal{E}_{\geq t'}\right]' \cap \mathcal{E}_{\geq t} \simeq B(\mathcal{H}), \quad \text{whenever} \quad t' > t\,,$$

where $\mathcal{H} \simeq$ infinite-dim. separable Hilbert space.

We now suppose that, at an initial time $t_0 = 0$, the system S is prepared in a state without photons; i.e., the radiation field is in its so-called **vacuum state,** denoted by $\left|\emptyset\right\rangle\left\langle\emptyset\right|$. In the limit where $c \to \infty$, the state of S restricted to the algebra $\mathfrak{D}_{\geq t}$ is then given by the **vacuum state, for all times** t, because any photons present or produced at times $< t$ have escaped to spatial infinity at time t ($c = \infty$!). This fact, along with (6), allows us to *entirely neglect the degrees of freedom of the radiation field, provided S is prepared in the vacuum state of the radiation field.*

Let Ω_0 be the density matrix on $\mathfrak{H}_A$ describing the initial state of the atom at time $t_0 = 0$. Then the *"ensemble state" of S at time t* obtained by restriction of the state $\Omega_0 \otimes \left|\emptyset\right\rangle\left\langle\emptyset\right|$ to $\mathcal{E}_{\geq t}$ is given by

The time evolution of the state of the atom

$$\Omega_t \otimes \left|\emptyset\right\rangle\left\langle\emptyset\right|, \tag{7}$$

where Ω_t is a density matrix on $\mathfrak{H}_A$. It is straightforward to find the *evolution equation* for Ω_t. It is given by a **Lindblad equation**[2]

$$\dot{\Omega}_t = \mathfrak{L}_\alpha[\Omega_t], \quad \text{with}$$
$$\mathfrak{L}_\alpha[\Omega] := -i\hbar^{-1}\big[H_A, \Omega\big] + \alpha \sum_k \Big[V_k \Omega V_k^* - \frac{1}{2}\big\{\Omega, V_k^* V_k\big\}\Big], \tag{8}$$

where $\alpha = e^2$ is the *fine structure constant,* and the operators V_k, $k = 1, 2, \ldots$, can be calculated from the transition amplitudes $\big\{d_{ij} \,|\, i, j = 0, 1, \ldots, N-1\big\}$ and "form factors" of photon emission- and absorption operators appearing in the interaction Hamiltonian H_I.

Note: Entropy $S[\Omega_t]$ may (but need not) **increase** in t ($\longrightarrow$ dissipation)!

Eq. (8) is analogous to the diffusion equation (4) considered in Sect. 1. Just like **"unraveling"** of (4) yields the *Poisson jump process* describing a random walker whose Law is given in Eqs. (1)-(3), **"unraveling"** of Eq. (8) will yield the Law of what we call a *"quantum Poisson process"*, which describes the *stoch. dynamics of* **individual** *quantum systems* $\simeq S$.

[2] In the type-I case this is always true!

3. Lindblad dynamics and Quantum-Poisson processes

We consider a system $\simeq S$, as in Sect. 2, prepared at time $t_0 = 0$ in an initial state $\Omega_0 \otimes |\emptyset\rangle\langle\emptyset|$, where

$\Omega_0 := \Pi_0$ is a **pure** state, i.e., Π_0 is a rank-1 orthogonal projection,

and $|\emptyset\rangle$ is the **vacuum state** of the radiation field ($\nexists$ any photons), which we describe in the limiting regime $c \to \infty$. As in Sect. 2, we suppose that $\nexists$ any detectors recording photons.
I recall that the **"ensemble state"** of S at time t is then given by $\Omega_t \otimes |\emptyset\rangle\langle\emptyset|$, where Ω_t satisfies a *Lindblad equation*

$$\Omega_{t+dt} = \Omega_t + \mathfrak{L}_\alpha[\Omega_t]dt + \mathcal{O}(dt^2)\,, \quad \text{with}$$
$$\mathfrak{L}_\alpha[\Omega] := -i\hbar\big[H_A, \Omega\big] + \alpha \sum_k \Big[V_k \Omega V_k^* - \frac{1}{2}\big\{\Omega, V_k^* V_k\big\}\Big]\,, \ldots \tag{9}$$

Since, in the limit $c \to \infty$, the state of the radiation field is the vacuum state, $|\emptyset\rangle$, which is *pure*, at **all** times $t > 0$, we may neglect this field henceforth and think of S as just consisting of a single atom with dissipative dynamics given by (9). But, according to the *ETH - Approach*, the state of an **individual** atom isomorphic to S is **pure** at all times $t > 0$.

Diagonalizing Ω_{t+dt}

Thus, let Π_t be the pure state of an individual atom at time t. Then the state at time $t+dt$, when averaged over an ensemble of atoms, **all** identical to S and prepared in the **same** pure state Π_t at time t, is given by

$$\Omega_{t+dt} = \Pi_t + \mathfrak{L}_\alpha[\Pi_t]\, dt + \mathcal{O}(dt^2)\,. \tag{10}$$

The state Ω_{t+dt} isn't pure, anymore. Since $\Omega_{t+dt} = \Omega^*_{t+dt} > 0$, with $\mathrm{tr}\big(\Omega_{t+dt}\big) = \mathrm{tr}\,\Pi_t = 1$, the decomposition of Ω_{t+dt} into a *convex combination of pure states* takes the form (↗ spectral theorem)

$$\Omega_{t+dt} = p_{nj}[t, t+dt]\,\Pi^0_{t+dt} + \sum_{\delta=1,\ldots,N-1} p_\delta[t, t+dt]\,\Pi^\delta_{t+dt}\,, \tag{11}$$

where

$$\begin{gathered} p_{nj}[t, t+dt] \equiv p_0[t, t+dt] = 1 - \mathcal{O}(dt) > 0, \\ p_0 > p_1 \geq \cdots \geq p_{N-1} \geq 0, \quad \text{with} \quad p_\delta = \mathcal{O}(dt),\, \forall\, \delta \geq 1\,, \\ p_{nj}[t, t+dt] + \sum_{\delta=1,\ldots,N-1} p_\delta[t, t+dt] = 1\,, \quad (nj = \text{“no jump”})\,. \end{gathered}$$

The state-reduction postulate of the *ETH* - Approach

According to the *ETH* - Approach, **one** of the projections Π^{δ}_{t+dt}, $\delta = 0, 1, \ldots, N-1$, *randomly chosen,* is the state of an **individual** atom at time $t + dt$: According to **Born's Rule**, the probability/frequency that Π^{δ}_{t+dt} (for arb. δ) is chosen is given by $p_{\delta}[t, t+dt]$ (with $p_0 \equiv p_{nj}$). In order to come up with explicit expressions for these quantities, we apply *"infinitesimal (analytic) perturbation theory" (IPT)*, considering Π_t in (10) as an *unperturbed op.* (H_0), $\mathfrak{L}_{\alpha}[\Pi_t]$ as a *perturbation* (V), and $dt =$ strength of perturbation.

Digression on *IPT*: Let H_0 be an operator on $\mathbb{C}^N$ with a simple eigenvalue E_0 separated from the rest of its spectrum by a strictly positive gap. We let $\Pi := |\psi_0\rangle\langle\psi_0|$ be the projection onto the eigenvector, ψ_0, of H_0 corresponding to the eigenvalue E_0. Let V be an operator on $\mathbb{C}^N$, and consider the perturbed op.

$$H(\varepsilon) := H_0 + \varepsilon\, V, \qquad \text{with} \quad \varepsilon \equiv dt \ll 1\,.$$

We are interested in formulae for the ev $E_0(\varepsilon)$ of $H(\varepsilon)$ in the vicinty of the unperturbed ev E_0 and the eigenprojection, $\Pi(\varepsilon)$, onto the eigenvector of $H(\varepsilon)$ corresp. to $E_0(\varepsilon)$.

Infinitesimal (analytic) perturbation theory

We define an operator S by

$$S := \big(H_0 - E_0\big)^{-1}\Pi^{\perp}\cdot V\cdot\Pi + \Pi\cdot V\cdot\Pi^{\perp}\big(E_0 - H_0\big)^{-1}. \qquad (12)$$

Theorem *IPT*: *The ev* $E_0(\varepsilon)$ *and the eigenprojection* $\Pi(\varepsilon)$ *are given by*

$$\begin{aligned} E_0(\varepsilon) &= E_0 + \varepsilon\cdot\mathrm{tr}\big(\Pi\cdot V\big) + \mathcal{O}(\varepsilon^2) \\ \Pi(\varepsilon) &= \Pi - \varepsilon\big[S,\Pi\big] + \mathcal{O}(\varepsilon^2)\,. \end{aligned} \qquad (13)$$

Up to errors of order $\mathcal{O}(\varepsilon^2)$*, the remaining eigenvalues of* $H(\varepsilon)$ *can be found by diagonalizing the matrix* $\Pi^{\perp}\cdot H(\varepsilon)\cdot\Pi^{\perp}$.

The equation for $E(\varepsilon)$ is called *Feynman-Hellmann thm.*, the one for $\Pi(\varepsilon)$ is *first-order* (Schrödinger) *perturbation theory.*

Remark: Given a differentiable family, $\big\{H_t \,\big|\, 0\le t\le 1\big\}$, of operators with the property that the spectrum and the eigenvectors of H_0 are known explicitly, one can find the spectrum and the eigenvectors of $H_t, 0 < t \le 1$, by solving an initial value problem consisting of a system of ordinary differential equations derived from formula (13); (this very useful variant of perturbation theory is really what I call *IPT*).

IPT and the *stochastic, non-linear* evolution of *individual* systems

We use Theorem *IPT* to derive from (10) a system of ordinary differential equations for the quantities, $p_\delta[t, t+dt]$ and Π^δ_{t+dt} (see R.S. of Eq. (11)), setting $H_0 := \Pi_t$, $V := \mathfrak{L}_\alpha[\Pi_t]$, $\varepsilon := dt$ (small). The spectrum of H_0 is given by $\{1, \underbrace{0, \ldots, 0}_{N-1 \, times}\}$, the ev 1 being simple. By (12)

$$S \equiv S_t := -\Pi_t^\perp \cdot \mathfrak{L}_\alpha[\Pi_t] \cdot \Pi_t + \Pi_t \cdot \mathfrak{L}_\alpha[\Pi_t] \cdot \Pi_t^\perp \,, \tag{14}$$

which is a well-defined operator on $\mathbb{C}^N$. Applying Theorem *IPT* we find the following system of differential equations:

(I) Feynman-Hellmann $\Rightarrow$

$$p_{nj}[t, t+dt] = 1 + \mathrm{tr}\big(\Pi_t \cdot \mathfrak{L}_\alpha[\Pi_t]\big)\, dt + \mathcal{O}(dt^2), \text{ hence}$$

$$\frac{\ell \mathrm{n}\, p_{nj}[t, t+dt]}{dt} = \mathrm{tr}\big(\Pi_t \cdot \mathfrak{L}_\alpha[\Pi_t]\big) < 0 \;\; \Rightarrow$$

$$\boxed{p_{nj}[0, t] = \exp\Big\{ \int_0^t \mathrm{tr}\big(\Pi_s \cdot \mathfrak{L}_\alpha[\Pi_s]\big) ds \Big\} < 1}$$

Time-evolution of state in absence of "quantum jumps"

(II) Time-dependence of state, Π_t, in the absence of quantum jumps, (i.e., eigenprojection corresp. to ev $p_{nj}[t, t+dt]$ is chosen in an open interval of times containing t): Eq. (13) yields diff. eq.

$$\boxed{\frac{d\Pi_t}{dt} = \Pi_t^{\perp} \cdot \mathfrak{L}_{\alpha}[\Pi_t] \cdot \Pi_t + h.c.}$$

This is a system of **non-linear** (cubic) differential eqs. for Π_t.

(III) "Quantum jumps": The **spectrum** of the **non-negative** matrix $\Pi_t^{\perp} \cdot \mathfrak{L}_{\alpha}[\Pi_t] \cdot \Pi_t^{\perp}$ is given by $\left\{ \frac{p_{\delta}[t,t+dt]}{dt} \middle| \delta = 1, 2, \ldots, N-1 \right\}$. Hence

$$0 < p_{\delta}[t,t+dt] = \mathcal{O}(dt),$$

$$p_{nj}[t, t+dt] + \sum_{\delta=1}^{N-1} p_{\delta}[t, t+dt] = 1.$$

(IV) For $\alpha = 0$ (i.e., atom decoupled from radiation field), one finds that

$$p_{nj}[0, t] = 1 \quad \leftrightarrow \text{ no quantum jumps (!)}, \qquad \frac{d\Pi_t}{dt} = -i\big[H_A, \Pi_t\big],$$

$\rightarrow$ unitary evolution – as expected!

State trajectories with "quantum jumps"

We now introduce an analogue of the *Wiener measure* of BM, more precisely of the measure (3) in Sect.1. We suppose that the state of a system $\simeq S$ exhibits "quantum jumps" at times $t_1 < \cdots < t_n < t_{fin}$, with $t_1 > 0, t_{n+1} \equiv t_{fin}$. The state evolves continuously between jumps at times t_j and t_{j+1} according to the differential equations derived in items (I) and (II); it is denoted by $\Pi_t^{\delta_j|\delta_{j-1}}$, with $t_j < t < t_{j+1}$, where $t_0 = 0$, $\Pi_{t=0}^{\delta_0|\delta_{-1}} = \Pi_0$ is the initial state of the system. We propose to calculate the probability of a trajectory, $\mathfrak{T}_n$, of states of an individual system $\simeq S$ given by

$$\mathfrak{T}_n := \left\{ \Pi_t^{\delta_j|\delta_{j-1}} \middle| t_j < t < t_{j+1}, j = 0, 1, \ldots, n \right\}, \tag{15}$$

where $\Pi_{t_j}^{\delta_j|\delta_{j-1}}$ is the eigenprojection corresponding to the eigenvalue $\frac{p_{\delta_j}[t_j, t_j+dt_j]}{dt_j}$ of the non-negative matrix

$$\left(\Pi_{t_j}^{\delta_{j-1}|\delta_{j-2}}\right)^{\perp} \cdot \mathfrak{L}_\alpha\left[\Pi_{t_j}^{\delta_{j-1}|\delta_{j-2}}\right] \cdot \left(\Pi_{t_j}^{\delta_{j-1}|\delta_{j-2}}\right)^{\perp}.$$

Following (I), we define

$$p_{nj}^{\delta_j|\delta_{j-1}}[t_j, t_{j+1}] := \exp\left\{ \int_{t_j}^{t_{j+1}} \mathrm{tr}\left(\Pi_t^{\delta_j|\delta_{j-1}} \cdot \mathfrak{L}_\alpha\left[\Pi_t^{\delta_j|\delta_{j-1}}\right] \right) dt \right\}.$$

A quantum-mechanical analogue of the Wiener measure

The probability, $W_{\mathfrak{T}_n}$, of the trajectory $\mathfrak{T}_n$ introduced in Eq. (15) is then given by

$$W_{\mathfrak{T}_n}\big[\delta_1, t_1, \ldots, \delta_n, t_n\big] \prod_{j=1}^{n} dt_j :=$$

$$= \Big\{ \prod_{j=0}^{n-1} p_{nj}^{\delta_j|\delta_{j-1}}[t_j, t_{j+1}] p_{\delta_{j+1}}[t_j, t_{j+1} + dt_{j+1}] \Big\} p_{nj}^{\delta_n, \delta_{n-1}}[t_n, t_{fin}] \,. \quad (16)$$

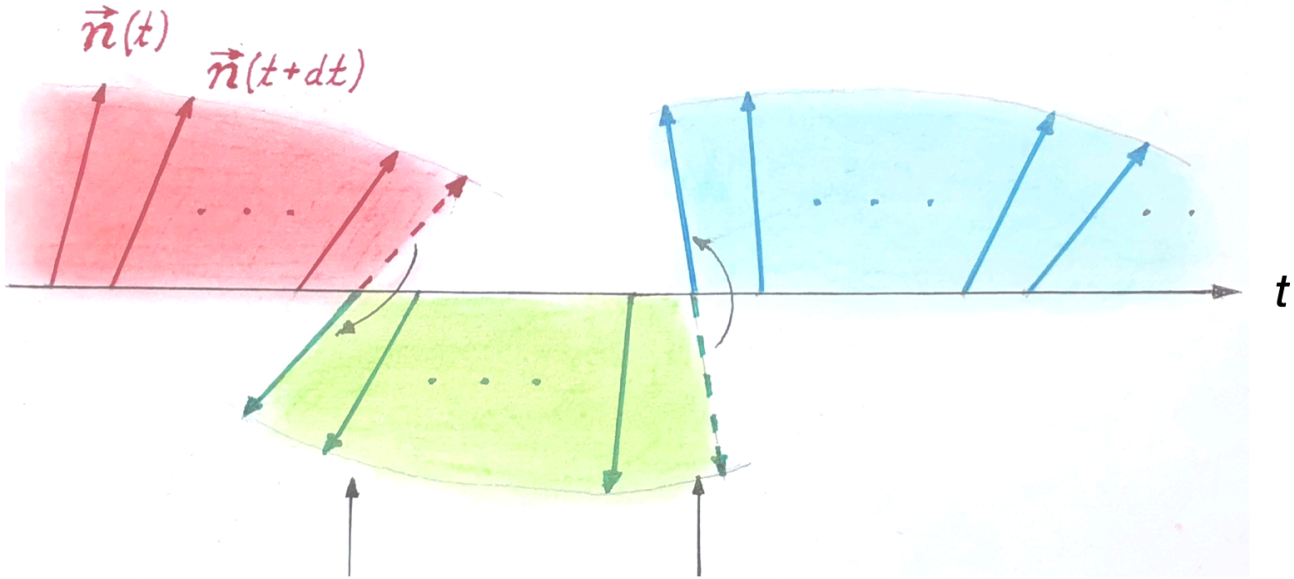
This formula serves to determine the probability of "measureable sets" of quantum trajectories, $\mathfrak{T}$, of **individual** systems $\simeq S$ with an arbitrary number of "quantum jumps" in the time interval $[0, t_{fin}]$.

Remark: If an average, $\mathbb{E}$, of the states in the trajectories $\mathfrak{T}_n$ is taken using the "measures" $W_{\mathfrak{T}_n}$, with $n = 0, 1, 2, \ldots$, then we recover the ensembles states Ω_t that obey the Lindblad equation (10); i.e.,

$$\sum_n \mathbb{E}\big[\mathfrak{T}_n\big] = \big\{ \Omega_t \,\big|\, 0 \leq t \leq t_{fin},\, \Omega_0 = \Pi_0 \big\} \,.$$

4. The example of a two-level atom – fluorescence

A pictorial representation of quantum trajectories of states (denoted here by $\vec{n}_t$) of **individual** systems $\simeq S$:



quantum jumps

In the following, we consider a simple concrete example. The atom has only **two** internal states, i.e., $\mathfrak{H}_A = \mathbb{C}^2$, and its Hamiltonian is given by

$$H_A := \hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{17}$$

The time evolution of ensemble states

The states of the atom are described by 2×2 matrices of the form

$$\Omega \equiv \Omega(\vec{n}) := \frac{1}{2}\big[\mathbf{1}_2 + \vec{n} \cdot \vec{\sigma}\big], \quad \vec{n} \in \mathbb{R}^3, \text{ with } |\vec{n}| \leq 1, \tag{18}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) =$ vector of Pauli matrices. The state $\Omega(\vec{n})$ is **pure** iff $\vec{n}$ is a unit vector, i.e., $\vec{n}$ lies on the so-called *Bloch sphere;* and

$$\Omega(\vec{n}) + \Omega(-\vec{n}) = \mathbf{1}_2 .$$

We introduce lowering- and raising operators

$$\sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ resp.}$$

In order to describe the *fluorescence* of such an atom, one sets

$$\mathfrak{L}_\alpha[\Omega] := -i\hbar^{-1}\big[H_A, \Omega\big] + \alpha\big[\sigma_- \Omega\, \sigma_+ - \frac{1}{2}\big\{\Omega, \sigma_+\sigma_-\big\}\big] . \tag{19}$$

The Lindblad equation can then be written as a **linear equation** for the vector $\vec{n}$ in the unit ball of $\mathbb{R}^3$:

Unraveling the Lindblad evolution

$$\dot{\vec{n}}(t) = \underbrace{\frac{\omega}{2}\,\vec{e}_3 \wedge \vec{n}(t)}_{\text{precession around } \vec{e}_3} - \underbrace{\frac{\alpha}{4}\Big[2\vec{e}_3 + \vec{n}(t) + n_3(t)\cdot\vec{e}_3\Big]}_{\text{dissipation}}. \qquad (20)$$

We observe that $\vec{n}(t) \to -\vec{e}_3$, as $t \to \infty$.
We now "unravel" Eq. (20) by specializing the procedure of Sect. 3 to the present model: Let $\vec{n}(t)$ be a unit vector, and let $\bar{\vec{n}}(t+dt)$ be given by

$$\bar{\vec{n}}(t+dt) = \vec{n}(t) + \Big\{\frac{\omega}{2}\,\vec{e}_3 \wedge \vec{n}(t) - \frac{\alpha}{4}\Big[2\vec{e}_3 + \vec{n}(t) + n_3(t)\cdot\vec{e}_3\Big]\Big\}dt\,.$$

The prinicples of the *ETH - Approach* then imply that $\bar{\vec{n}}(t+dt)$ must be replaced by a unit vector $\vec{n}(t+dt)$ whose Law is given by (see figure!)

$$\begin{aligned}
\vec{n}(t+dt) &= \frac{\bar{\vec{n}}(t+dt)}{\big|\bar{\vec{n}}(t+dt)\big|}, \qquad \text{with probability} \quad p_{nj}[t, t+dt], \\
\vec{n}(t+dt) &= -\frac{\bar{\vec{n}}(t+dt)}{\big|\bar{\vec{n}}(t+dt)\big|}, \qquad \text{with probability} \quad p_{flip}[t, t+dt]\,,
\end{aligned} \qquad (21)$$

The Poisson flip process on the Bloch sphere

where

$$p_{nj}[t, t+dt] = \frac{1 + \left|\vec{\overline{n}}(t+dt)\right|}{2} = 1 - \mathcal{O}(dt)$$
$$p_{flip}[t, t+dt] = \frac{1 - \left|\vec{\overline{n}}(t+dt)\right|}{2} = \mathcal{O}(dt)\,, \tag{22}$$

hence $p_{nj}[t, t+dt] + p_{flip}[t, t+dt] = 1$, as required.

Theorem: *Suppose there isn't any flip in the time interval* $[t_1, t_2)$*, with* $\vec{n}(t = t_1)$ *a given unit vector. Then, for* $t \in [t_1, t_2)$*,* $\vec{n}(t)$ *has the form*

$$\vec{n}(t) = \begin{pmatrix} \sqrt{1 - n_3(t)^2}\cos(\omega t + \gamma) \\ \sqrt{1 - n_3(t)^2}\sin(\omega t + \gamma) \\ n_3(t) \end{pmatrix}\,,$$

for some constant γ*, where* $n_3(t)$ *solves the* **non-linear** *equation*

$$\dot{n}_3(t) = -\frac{\alpha}{4}\Big(1 + n_3(t)\Big)\Big(1 - n_3(t)\Big)\Big(2 + n_3(t)\Big)\,. \tag{23}$$

Consequences of (23), and formulae for p_{nj}

- If $n_3(t_1) = -1$, i.e., atom in *ground-state,* then $n_3(t) \equiv -1$, and

$$p_{nj}[t_1, t] \equiv 1, \; \forall \; t > t_1 \, .$$

- If $n_3(t_1) = +1$, i.e., atom in *excited state,* then $n_3(t) = +1$, for $t \in [t_1, t_2)$, followed by transition to *ground-st.* at time t_2, with

$$p_{nj}[t_1, t_2) = \exp[-\alpha \, (t_2 - t_1)] \qquad \text{(exponential decay law)} \, .$$

- If $n_3(t_1) \in (-1, 1)$ then the state variable $\vec{n}(t)$ of the atom precesses around $\vec{e}_3$ with angular frequency ω, and

$$\frac{d\ell n\big(p_{nj}[t_1, t]\big)}{dt} = -\frac{\alpha}{4}\Big(1 + n_3(t)\Big)^2 ,$$

whose solution is given by

$$p_{nj}[t_1, t] = \exp[-\lambda(t)], \text{ where } \lambda(t) = \frac{1}{2}\int_{n_3(t)}^{n_3(t_1)} \frac{1+\tau}{(1-\tau)(2+\tau)} d\tau \, .$$

Conclusions

Using these (very explicit) results and (22), state trajectories with flips at times $t_n + dt_n, n = 1, 2, \ldots$, can be treated as in Sect. 3.

- For sufficiently large times (after finitely many flips), the atom is always found in its ground-state; i.e., $\vec{n}(t) \longrightarrow -\vec{e}_3$, as $t \longrightarrow \infty$.

—

Let's conclude: In idealized models of systems of charged matter interacting with the quantized radiation field, described in the limit where the velocity of light $c \longrightarrow \infty$, the principles of the *ETH - Approach* to (or the *"ETH - Completion"* of) Quantum Mechanics apparently yield **totally explicit predictions** concerning the **stochastic time evolution** of states of individual systems. This evolution is given by a new type of stochastic process, called *quantum Poisson process*, which I have tried to describe in this talk.

It would be interesting to test the fine-print of these predictions in experiments. The math of *quantum Poisson processes* (& generalizations) deserves to be worked out more fully. –
A new (dynamical) approach to *"quantum chaos"* is emerging.

Thanks for listening!