Quantum Dynamics of Systems Under Repeated Observation[1]

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Outline

I propose to study the effective quantum dynamics of systems under repeated observation, more specifically ones interacting with a chain of independent probes, which, afterwards, are subject to a projective measurement and are then lost.

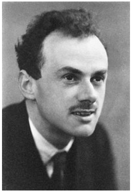
This leads to a theory of indirect measurements of time-independent quantities (non-demolition measurements).

Subsequently, a theory of indirect weak measurements of time-dependent quantities is outlined, and a new family of diffusion processes, dubbed *quantum jump processes*, is described. –

To conclude, some open problems are sketched.

Here are the founding fathers of Quantum Mechanics:




Credits and Contents

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1. Introduction – some fundamental questions and claims

In our courses, we tend to describe the QM of systems, S, in terms of pairs of a *Hilbert space*, $\mathcal{H}_S$, of pure state vectors and a *unitary propagator*, $(U(t,s))_{t,s\in\mathbb{R}}$, descr. the "time evolution" of its states.

Unfortunately, these data hardly encode *any* information about S that would enable one to draw conclusions about its physical properties, and they give the erroneous impression that quantum theory might be a *linear and deterministic theory*.

$\rightarrow$ Fundamental questions and problems:

1. What do we have to add to the usual formalism of Quantum Mechanics (QM) to arrive at a mathematical structure that – through interpretation – can be given unambiguous physical meaning; hopefully *without the intervention* of "observers"?
2. Where does the *intrinsic randomness* of QM come from, given the deterministic character of the Schrödinger and Heisenberg equations? How does it differ from classical randomness?

Fundamental questions

3. What do we mean by an *isolated* (but open) *system* in QM, and why is this an important notion? How can one prepare an isolated system in a *prescribed state*?
4. What is the meaning of *"observables"/phys. quantities* and of *states* of systems in QM? What is the *time evolution* of *phys. quantities* and of *states* in the **Heisenberg picture**? What is the role of the Schrödinger equation in all this? Does *"wave-function collapse"* occur, and why?
5. What is a *potential/actual event*, and how should one describe an *instrument* used to record an event, in QM?

There are too many examples[2] of simple experimental situations where we do not really understand how to apply quantum theory in a logically coherent way to describe what one sees in experiments. Almost 100 years after the discovery of matrix mechanics, this is an intellectual scandal!

[2]e.g., a fluorescent atom put into a stationary laser beam inducing stochastic emission of photons by the atom

Metaphor for the "mysterious holistic aspects" of QM

QM is QM-as-QM and everything else is everything else

"The one thing to say about art is that it is one thing. Art is art-as-art and everything else is everything else." (Ad Reinhardt)

It is time to open this black box and see what's inside!

2. Examples of systems under repeated observation – Haroche-Raimond- & solid-state exps., particle tracks

The *ETH approach* represents a *"quantum theory without observers"* describing *actual events* and their observation in projective measurements, using *instruments*. – Taking this (or another satisfactory) theory of projective measurements for granted, the theory of *indirect* (in particular, non-demolition-) *measurements* is fairly straightforward and can be presented with perfect mathematical precision.

The analysis of some simple examples of the *Theory of Indirect Measurements* of physical quantities – pioneered by Karl Kraus – is the *main topic of this lecture.*

Karl Kraus (1938-1988)

A metaphor for the meaning of *indirect observations*

Plato's Allegory of the Cave – *'Politeia'*, in: Plato's *'Republic'*

As Plato was anticipating, more than 350 years BC, all we *"prisoners of our senses"* are able to perceive of the world are "shadows of reality" – in the form of long streams of crude, uninteresting, directly perceptible signals (= outcomes of projective measurements) – from which meaningful facts can then be **reconstructed**.

As *Socrates* explains: Philosophers (= mathematicians and theor. physicists) are "liberated prisoners" who are able to infer the fabric of reality from the shadows it creates on the wall of the cave.

Systems/experiments to be studied in this talk

I. The *Haroche-Raimond experiment:* $S = E \vee C$ (cavity)

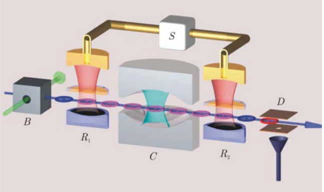
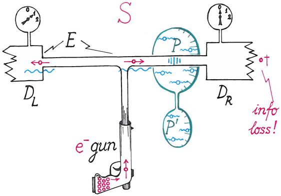


Fig. 4: Experimental setup to study microwave field states with the help of circular Rydberg atoms (see text).

B: atom/probe gun, R_1: State prep., C: Cavity, ..., D: Detector

II. A *solid-state (Gedanken-) experiment:* $S = E \vee P$



Details concerning experiments I and II

Isolated open system: $S = E \vee P$, where P = subsystem of interest, i.e., cavity C, **or** quantum dot; E = "environment/equipment" consisting of:

(1) Probes: Independent atoms $A_1, A_2, \dots$ prepared in R_1, [**or** indep. electrons prepared in e^-gun] – all in the same initial state.

In time interval $[(m-1)\tau, m\tau)$, m^{th} atom streams through cavity C, [**or** m^{th} e^- travels from e^-gun through T-shaped wire to either the detector D_L, or the detector D_R, respectively].

τ: duration of a "measurement cycle."

(2) An atom detector, D, [**or** two electron detectors, D_L, D_R, resp.] serving to perform projective measurements on probes.

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It is a little easier to picture how the *solid-state experiment* works:

- Physical quantity referring to the quantum dot P to be measured: Charge of P, rep. by operator, $\mathcal{N}$, with $\text{spec}(\mathcal{N}) = \{0, 1, \dots, N\}$.
- Physical quantities referring to environment E, i.e., the electrons:

$$\{\mathbf{1}_P \otimes \mathbf{1}_{e_1^-} \otimes \cdots \otimes X_{e_m^-} \otimes \mathbf{1}_{e_{m+1}^-} \otimes \dots\}_{m=1,2,3,\dots},$$

Description of solid-state experiment

where the operator $X_{e_m^-}$ acts on the one-particle Hilbert space of the m^{th} electron traveling through the T - shaped wires towards D_L, D_R, resp. It is given by

$$X_{e_m^-} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

with infinitely degenerate eigenvalues $\xi \in \mathcal{X}_S = \{-1, +1\}$:

$$\xi = +1 \leftrightarrow e_m^- \text{ hits } D_L, \qquad \xi = -1 \leftrightarrow e_m^- \text{ hits } D_R.$$

From now on, "L" is usually replaced by $+1$, and "R" by -1. The *eigenprojection* of $X_{e_m^-}$ corresp. to the eigenvalue ξ is denoted by π_ξ^m; The quantity $X_{e_m^-}$ is measured around the time $m \cdot \tau$.

In the following, ρ ($=$ some density matrix) denotes the state of S.

Our aim is to determine the *probability*, μ_ρ, *of the events*, $\{\xi_1, \ldots, \xi_k\}$, *that, for* $m = 1, 2, ..., k$, *the* m^{th} *electron hits the detector* D_{ξ_m}, $\xi_m = \pm 1$; $k = 1, 2, ...$.

The LSW formula

For (strictly) independent electrons [3], this probability is given by a formula proposed by *Lüders, Schwinger* and *Wigner* (LSW):

$$\mu_\rho(\xi_1, \xi_2, \ldots, \xi_k) := \mathrm{tr}\left(\pi^k_{\xi_k} \cdots \pi^1_{\xi_1}\, \rho\, \pi^1_{\xi_1} \cdots \pi^k_{\xi_k}\right) \tag{1}$$

Since $\pi^k_1 + \pi^k_{-1} = \mathbf{1}, \forall k$, and because of the cyclicity of the trace,

$$\sum_{\xi_k} \mu_\rho(\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_k) = \mu_\rho(\xi_1, \xi_2, \ldots, \xi_{k-1}).$$

Thus, by a lemma due to *Kolmogorov*, μ_ρ extends to a measure on the space, Ξ, of "histories" ($= \infty$ long measurement records, $\underline{\xi} := \left(\xi_j\right)_{j=1}^{\infty}$), equipped with the σ-algebra, Σ, generated by cylinder sets.

The measure μ_ρ can be decomposed into a convex combination of *"extremal" measures*:

$$\mu_\rho(\underline{\xi}) = \int_{\Xi_\infty} \mathrm{d}P_\rho(\nu)\mu(\underline{\xi}|\nu), \tag{2}$$

[3] the property of strict indep. of e^-'s is a special case of *"decoherence"!*

Exchangeable probability measures

where Ξ_∞ is the spectrum of the algebra of bounded functions on Ξ *measureable at* ∞,[4] (ν denotes points in Ξ_∞), Ξ_∞ is the *"space of facts"* (or of the "Dinge an sich" – quite in the sense of *Plato* and *Kant*).

First, we consider the situation where the e^-'s are indep., and their passage from the electron gun through the T - shaped wire to one of the detectors D_ξ, $\xi = \pm 1$, does *not* affect the charge, ν, of the quantum dot P, assumed to be *conserved* $\rightarrow$ *"non-demolition measurements"*. One can then argue that the measure μ_ρ is *exchangeable*, i.e.:

$$\mu_\rho\big(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}\big) = \mu_\rho\big(\xi_1, \ldots, \xi_k\big),$$

for all permutations, σ, of $\{1, \ldots, k\}$, for arbitrary $k < \infty$. According to *De Finetti's* Theorem this implies that (in Eq. (2))

$$\mu(\underline{\xi}_k|\nu) = \prod_{m=1}^{k} p(\xi_m|\nu), \quad \underline{\xi}_k := (\xi_1, \ldots, \xi_k), \quad \forall k \in \mathbb{N}. \qquad (3)$$

[4] equiv. classes (w.r. to a measure class determined by normal states of S) of functions on Ξ **not** dep. on any finite number of measurement outcomes!

Interpretation of Ξ_∞ in the solid-state experiment

Suppose every electron traveling from the e^--gun to one of the detectors $D_{\pm 1}$ is prepared in *same* one-particle state ϕ_0. Assuming that the charge operator, $\mathcal{N}$, of the quantum dot P is a conservation law, the time evol. of the state ϕ_0 during one measurement cycle is given by

$$U_\nu \phi_0,$$

where U_ν is a unitary operator on the one-electron Hilbert space dep. on the charge $\nu \leq N$ of P: The charge ($\propto$ nb. of e^-) bound by P creates a "Coulomb blockade" in the right arm of the T - shaped wire; whence: *the larger ν, the more likely it is that an electron in the wire will be scattered onto the detector $D_1 \equiv D_L$.*

The projection onto one-electron wave functions that vanish identically near $D_{-\xi}$ is denoted by π_ξ. The probability, $p(\xi|\nu)$, that an e^- hits D_ξ is given by *Born's Rule*

$$p(\xi|\nu) = \langle \phi_0, U_\nu^* \pi_\xi U_\nu \, \phi_0 \rangle, \tag{4}$$

and the space Ξ_∞ of the "Dinge an sich" is given by

$$\Xi_\infty = \text{spec}(\mathcal{N}) = \{0, 1, 2, \dots, N\}, N < \infty, \quad \mathcal{N} = \text{charge operator of } P.$$

3. Indirect Non-Demolition Measurements: Basic Assumptions and General Results

Thinking of the solid-state experiment, we will henceforth assume:[5]

(i) The measures μ_ρ are exchangeable (non-demolition observations involving independent e^-!) $\Rightarrow$ they are convex combinations of product measures

$$\mu(\underline{\xi}_k|\nu) = \prod_{m=1}^{k} p(\xi_m|\nu), \quad \xi_m \in \mathcal{X}_S, \forall m, \ |\mathcal{X}_S| < \infty, \ \nu \in \Xi_\infty.$$

(ii) The space of "facts" is a finite set of points (charge values):

$$\Xi_\infty = \{0, 1, 2, \ldots, N\}, \qquad \text{for some } \ N < \infty. \qquad (6)$$

(iii) It is assumed that $p(\xi|\cdot)$ separates points of Ξ_∞: There exists $\kappa > 0$ such that

$$\min_{\nu_1 \neq \nu_2} |p(\xi|\nu_1) - p(\xi|\nu_2)| \geq \kappa > 0, \quad \text{for some } \xi \in \mathcal{X}_S. \quad (7)$$

[5] these assumptions can and have been generalized

Summary of main results

Equivalence classes of functions on the space Ξ of histories measureable at ∞ form an abelian algebra: the *algebra of "observables at infinity"*, ($=$ funs. on the "space of facts" Ξ_∞), which is isomorphic to $\text{Diag}_{(N+1)}$. An example of an "observable at infinity" is the *"asymptotic frequency"* of an event $\xi \in \mathcal{X}_S$: We define the *frequencies*

$$f_\xi^{(l,l+k)}(\underline{\xi}) := \frac{1}{k}\left(\sum_{m=l+1}^{l+k} \delta_{\xi,\xi_m}\right), \quad \text{with } \sum_{\xi} f_\xi^{(l,l+k)}(\underline{\xi}) = 1. \qquad (8)$$

Summary of Main Results:

(I) *Law of Large Numbers for exchangeable measures:* For every $\underline{\xi} \in \Xi$, the asymptotic frequency satisfies

$$\lim_{k\to\infty} f_\xi^{(l,l+k)}(\underline{\xi}) =: p(\xi|\nu), \qquad (9)$$

for some "fact" $\nu \in \Xi_\infty$.

"q-hypothesis testing" /parameter estimation

Definition: With each $\nu \in \Xi_\infty$ we associate a subset, Ξ_ν of Ξ def. by

$$\Xi_\nu(l,k;\underline{\varepsilon}) := \{\underline{\xi} \mid |f_{\underline{\xi}}^{(l,l+k)}(\xi) - p(\xi|\nu)| < \varepsilon_k\}, \qquad (10)$$

where

$$\varepsilon_k \to 0, \ \sqrt{k}\,\varepsilon_k \to \infty, \quad \text{as } k \to \infty$$

(II) *Distinguishability:* It follows from Hyp. (7) and definition (8) that, for k so large that $\varepsilon_k < \kappa/2$,

$$\Xi_{\nu_1}(l,k;\underline{\varepsilon}) \cap \Xi_{\nu_2}(l,k;\underline{\varepsilon}) = \emptyset, \quad \nu_1 \neq \nu_2.$$

(III) *Central Limit Theorem:* $\Rightarrow$ Under suitable hypotheses on the states ρ, e.g., (i) through (iii),

$$\mu_\rho\left(\bigcup_\nu \Xi_\nu(l,k;\underline{\varepsilon})\right) \to 1, \quad \text{as } k \to \infty.$$

hypothesis testing – ctd.

(I), (II) & (III) $\Rightarrow$ As $k \to \infty$, every measurement record $\underline{\xi}_k$ dets. a unique point (charge) $\nu \in \Xi_\infty$; (with error $\to 0$, as $k \to \infty$).

Moreover, *Born's Rule* holds: $\boxed{\mu_\rho\big(\Xi_\nu(I, k; \underline{\varepsilon})\big) \underset{k\to\infty}{\to} \rho(\delta_{\mathcal{N},\nu}) = P_\rho(\nu)}$

(See Eq. (3).)

(IV) *Theorem of Boltzmann-Sanov* $\Rightarrow$ If the measures μ_ρ are exchangeable one has that

$$\mu\big(\Xi_{\nu_1}(I, k; \underline{\varepsilon})|\nu_2\big) \leq C\ e^{-k\sigma(\nu_1\|\nu_2)},$$

where σ is the relative entropy of the distribution $p(\cdot|\nu_1)$ given $p(\cdot|\nu_2)$.

(V) *Theorem of Maassen-Kümmerer & Bauer-Bernard* (see (III), (IV), above!) $\Rightarrow$ In the *Haroche-Raimond exp.*, state of S, restr. to $B(\mathcal{H}_P)$, approaches a state, ρ^ν, with a *fixed number*, ν, *of photons* in the cavity P ($\equiv C$), as $k \to \infty$: *"Purification"!*
(Analogous results for solid-state experiment.)

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Summary of theory of non-demolition experiments

The theory of indirect measurements (of conserved quantities) outlined so far only concerns measurements of time-independent "facts", which correspond to points in Ξ_∞: *non-demolition measurements!* The outcomes of such measurements only depend on the tails of histories (at arb. late times). The "extremal" measures $\mu(\cdot|\nu)$, $\nu \in \Xi_\infty$, come from normal states ρ_ν. (This is a non-trivial statement.)

However, most interesting "facts" depend on time, i.e., are "events" appearing and disappearing, and $\Xi_\infty = \emptyset$! Thus, we must ask how one can infer or reconstruct information concerning *events* and their time evolution from finitely long records of projective measurements of quantities referring to probes and represented by operators that act on the Hilbert spaces of probes.

This question will be answered next!

4. Weak Measurements of Time-Dependent Quantities –

Markov Jump Processes on the Spectra of "Observables"

We consider an isolated physical system $S = P \vee E$, as before.
States of S are given by density matrices, ρ_S, acting on a Hilbert space $\mathcal{H}_S = \mathcal{H}_P \otimes \mathcal{H}_E$, where $\mathcal{H}_P = \mathbb{C}^{N+1}$, for some $N < \infty$. When restricted to observables of P, states are given by density matrices $\rho_P := \operatorname{tr}_E \rho_S$.

- Hilbert space of a single probe A_j: $\mathcal{H}_{A_j} \simeq \mathcal{H}_A$
- Initial state of each probe A_j: $\phi_0 \in \mathcal{H}_A$.
- Reference state in $\mathcal{H}_E$: $\bigotimes_{j=1}^{\infty} \phi_0^{(j)}, \quad \phi_0^{(j)} = \phi_0, \forall j$.
  Space $\mathcal{H}_E$ = completion of linear span of vectors $\bigotimes_{j=1}^{\infty} \psi^{(j)}$, with $\psi^{(j)} = \phi_0$, except for finitely many j.
- For each probe A_j, the same observable, represented by an operator

$$X = \sum_{\xi \in \mathcal{X}_S} \xi \, \pi_\xi, \qquad |\mathcal{X}_S| = k < \infty, \tag{11}$$

acting on $\mathcal{H}_{A_j}$, is measured in a detector D at a *random time* t_j, with $t_j < t_{j+1}, \ t_{j+1} - t_j$ Poissonian, $\forall j$; (D ignored in the following).

The formalism

During the j^{th} measurement cycle $(t_{j-1}, t_j]$, *only* A_j briefly interacts with P at time t_j. The measurement results for probes $A_1, \dots, A_{j-1}$ at times $\underline{t}_j = \left(t_k\right)_{k=1}^{j}$ are denoted by $\underline{\xi}_{j-1} = (\xi_k)_{k=1}^{j-1}$ (measurement record).

Notations:

- By $\rho_t^{(j-1)}(\underline{t}_{j-1}, \underline{\xi}_{j-1})$ we denote the *state* of P at time $t < t_j$, *after* interaction with probe A_{j-1} (at time t_{j-1}).
- Let $\mathcal{N}$ be an "observable" acting on $\mathcal{H}_P$ with simple spectrum,

$$\mathrm{spec}(\mathcal{N}) = \{0, 1, \dots, N\}, N < \infty.$$

By E_ν we denote the *spectral projection* of $\mathcal{N}$ corresp. to the ev ν.

We obtain a recursion formula for the states $\rho^{(j)} := \rho_{t_j}^{(j)}(\underline{t}_j, \underline{\xi}_j)$ of P:

$$\rho^{(j)} = \mathcal{Z}_{\xi_j}^{-1} V_{\xi_j} e^{-i(t_j - t_{j-1})H_P} \, \rho^{(j-1)} \, e^{i(t_j - t_{j-1})H_P} V_{\xi_j}^{*}, \tag{13}$$

where H_P is the Hamiltonian of P in the absence of interactions with probes, ...

Formalism – ctd.

... $\mathcal{Z}_\xi$ is a normalization factor, and the operator V_ξ is given by

$$V_\xi = \sum_\nu V_\xi(\nu), \qquad V_\xi(\nu) := E_\nu \sqrt{p(\xi|\nu)},$$

where

$$p(\xi|\nu) := \langle U_\nu \phi_0, \pi_\xi \, U_\nu \phi_0 \rangle,$$

with ϕ_0 the initial state of a probe, (see Eq. (4), Sect. 2).
Note that

$$V_\xi = V_\xi^*, \; [V_\xi, \mathcal{N}] = 0, \; \forall \xi, \quad \text{and} \quad \sum_{\xi' \in \mathcal{X}_S} V_{\xi'}^2 = \mathbf{1}. \tag{14}$$

The recursion formula (13) yields a trajectory of states of the subsystem P (the cavity/quantum dot) given by

$$\boxed{\rho_t(\underline{t}, \underline{\xi}) := e^{-i(t-t_j)H_P} \, \rho_{t_j}^{(j)}(\underline{t}_j, \underline{\xi}_j) \, e^{i(t-t_j)H_P}, \;\; t_j < t < t_{j+1}} \tag{15}$$

(see Kraus).

Averaged time-evolution of state of P

We suppose that the differences $t_j - t_{j-1}$ of times of interaction between the probes A_j and the subsystem P are Poisson distributed, with rate $\gamma = 1, \forall j$. Fixing a time t and taking an average, $\mathbb{E}$, over *measurement times* and *measurement outcomes*, we find that

$$\mathbb{E}\big[\rho_t(\underline{t}, \underline{\xi})\big] = e^{t\,\mathcal{L}}\,\rho, \tag{16}$$

where ρ is the initial state of the subsystem P at time $t = 0$, and $\mathcal{L}$ is a Lindblad generator given by

$$\mathcal{L}\,\rho = -i\,\mathrm{ad}_{H_P}(\rho) + \Big(\sum_{\xi \in \mathcal{X}_S} V_\xi\,\rho\,V_\xi\Big) - \rho. \tag{17}$$

Eq. (15) is what one calls an *"unravelling"* of the Lindblad evolution (16); it appears as the integrand in the *Dyson expansion* of the right side of (16), with the *second term* on the right side of (17) treated as the *perturbation*.

Main result

We suppose that the "Basic Assumption" (iii) of Sect. 3 is valid (i.e., that p separates points...). We assume furthermore that

$$H_P = \varepsilon h_p, \qquad \text{for some } \varepsilon > 0, \tag{19}$$

and we rescale time: $t =: \varepsilon^{-2}\tau$. We define a continuous-time Markov jump process, with state space = spec($\mathcal{N}$), paths $\nu_\tau(\omega)$, $\omega = (\underline{t}, \underline{\xi})$, and transition function generated by the Markov kernel:

$$Q(\nu, \nu') = -\frac{|\langle \nu | h_P | \nu' \rangle|^2}{\sum_{\xi \in \mathcal{X}_S} V_\xi(\nu) V_\xi(\nu') - 1} + cc, \quad \nu \neq \nu',$$

with $Q(\nu, \nu) = \cdots \geq 0, \forall \nu$.

We are now prepared to state our *Main Result*, (which has similarities with models illustrating the *"ETH approach"* to QM!).

Main result – ctd.

Theorem.

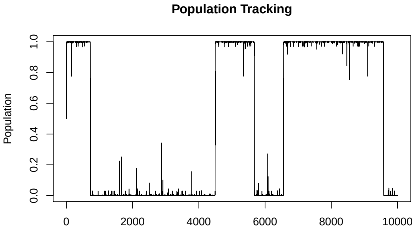
- *Convergence of qm evolution to Markov jump process:*

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\left[\rho_{\varepsilon^{-2}\tau}\left(\omega = (\underline{t}, \underline{\xi})\right)\right] = e^{-\tau Q}\rho_0,$$

  where $\rho_0 = \text{Diag}\,(\langle \nu | \rho | \nu \rangle)$.

- The state $\rho_{\varepsilon^{-2}\tau}\left(\omega = (\underline{t}, \underline{\xi})\right)$ approaches in law a diagonal matrix, $\text{Diag}\left(\delta_{\nu, \nu_\tau(\omega)}\right)$.

Numerical simulation for the behaviour of the diagonal matrix elements of $\rho_{\varepsilon^{-2}\tau}(\underline{t}, \underline{\xi})$ in the special case where $N = 1$ (i.e., $\mathcal{H}_P = \mathbb{C}^2$), for small ε:



5. Open Problems, Conclusions

- More general models of probes and "cavities"; in particular:
- Correlated probes; ∞-dimensional state spaces for cavity, P.
- More general models of indirect measurements of *time-dependent* quantities. –
  Approach to classical dynamics: Consider "observables," $\vec{\mathcal{N}}$, with continuous spectrum $\sigma(\vec{\mathcal{N}}) \simeq \mathbb{R}^d$; e.g., *particle position operators*. Then H_P may generate dynamics describing a particle motion on $\sigma(\vec{\mathcal{N}})$ resembling classical motion; the full dynamics of P may then describe tracks on $\sigma(\vec{\mathcal{N}})$ with "diffusive broadening:" Theory of *"Mott tracks"*; (now well understood in semi-classical regime – see next talk!). Etc.
- **Theory of projective (direct) measurements** – *ETH-Approach*

Our conclusion: Quantum Mechanics and its foundations are well and alive. There are plenty of beautiful new experiments testing fundamental aspects of Quantum Mechanics, and there are plenty of interesting problems for theorists to worry about – good luck!

Thank you!