Phase diagram of the Ashkin-Teller model on the square lattice

Yacine Aoun, joint work with Alexander Glazman and Moritz Dober
Université de Genève

Ben Safdie as Edward Teller and Cillian Murphy as Robert Oppenheimer in ’Oppenheimer’

- During their time in Los Alamos, they introduced and studied the model that later became known as the Ashkin-Teller model that we will study in this talk.

Statistics of Two-Dimensional Lattices with Four Components†

J. Ashkin and E. Teller
Pupin Physics Laboratories, Columbia University, New York, New York
(Received June 10, 1943)

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$$H^{AT}_{J_1,J_2,U}(\tau, \tau') = -\sum_{\substack{i,j \in \mathbb{Z}^2 \\ i \sim j}} J_1 \tau_i \tau_j + J_2 \tau_i' \tau_j' + U \tau_i \tau_j \tau_i' \tau_j',$$

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- We define $\beta^{\tau'}_c(J,U)$, $\beta^{\tau\tau'}_c(J,U)$ and $\gamma^{\tau'}, \gamma^{\tau\tau'}$ in the same fashion.

- Notice that by symmetry $\beta_c^{\tau}(J,U) = \beta_c^{\tau'}(J,U)$. Moreover, we have $\beta_c^{\tau\tau'} \leq \beta_c^{\tau}$, since by the GKS inequality

$$\mathbb{E}_{\beta,J,U}[\tau_0\tau_x]\mathbb{E}_{\beta,J,U}[\tau_0'\tau_x'] \leq \mathbb{E}_{\beta,J,U}[\tau_0\tau_0'\tau_x\tau_x'].$$

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Question: Do we always have $\beta_c^{\tau}(J,U) = \beta_c^{\tau\tau'}(J,U)$ or is it possible to have two different phase transitions ?

Beginning of story: physics' community

- In the early 70', there were several articles studying the AT model.

Two phase transitions in the Ashkin–Teller model†

F Y Wu‡ and K Y Lin

Department of Physics, National Tsing Hua University, Hsinchu, Taiwan, Republic of China

Duality relation between the Ashkin–Teller and the eight-vertex model

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Some Critical Properties of the Eight-Vertex Model*

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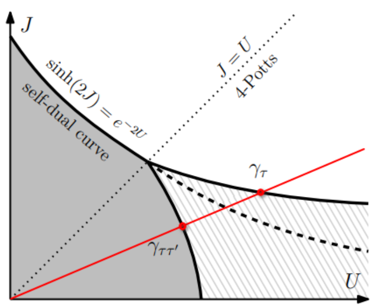
- In 1984, Pfister proved that there are two phase transitions when $2J < U$ by comparing to the Ising model with the use of the correlation inequalities.
- In 2023, we proved the conjecture.

Phase diagram of the symmetric Ashkin-Teller model

- In order to state our result, we firstly introduce the self-dual line: the set of (J, U) such that $\sin(2J) = e^{-2U}$. We define $\beta_{\rm sd}(J, U)$ as the unique value $\beta \geq 0$ such that $(\beta J, \beta U)$ lies on the self-dual line.



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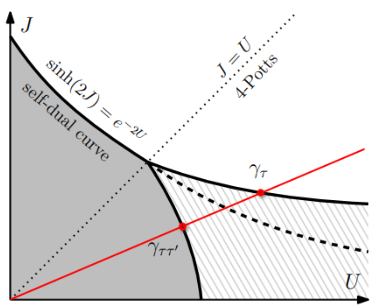

- Our main theorem is the following:

Theorem [A.–Dober-Glazman–2024]

Fix $J, U \geq 0$. If $J \geq U$, then $\beta_{\mathrm{sd}} = \beta_c^{\tau} = \beta_c^{\tau\tau'}$. If $U > J$, then $\beta_c^{\tau\tau'} < \beta_{\mathrm{sd}} < \beta_c^{\tau}$.

- Even though we don't know the precise location of the critical curves when $U > J$ (which has not even been conjectured in the physics' literature), we know that they are dual to each other in the following sense:

Theorem [A.–Dober-Glazman–2024]

Fix $J, U \geq 0$. $(J, U) \in \gamma^{\tau}$ if and only if $(J^*, U^*) \in \gamma^{\tau\tau'}$, where (J^*, U^*) are the images of (J, U) under the dual transformation.

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▶ We also prove the *sharpness* of the phase transition:

Theorem [A.–Dober-Glazman–2024]

Fix $U, J \geq 0$. The following is true

1. If $\beta < \beta_c^{\tau}$, then there exists $c > 0$ such that $\mathbb{E}_{\beta}[\tau_0 \tau_x] \leq e^{-c\|x\|}$.
2. If $\beta > \beta_c^{\tau}$, then there exists $C > 0$ such that $\mathbb{E}_{\beta,J,U}[\tau_0 \tau_x] > C$.

The same conclusion is true if we replace τ by $\tau\tau'$.

How we go about the proof of the main result ?

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- When $U > J > 0$, the reasoning is more complicated:

1. We firstly prove that there is exponential decay of the correlations in τ in the finite volume at the self-dual line, i.e. there exists $c > 0$ such that

$$\mathbb{E}_{\beta_{\mathrm{sd}},J,U,\Lambda_{|x|}}[\tau_0\tau_x] \leq e^{-c\|x\|}$$

(which implies that locally $\inf_{x\in\mathbb{Z}^d} \mathbb{E}_{\beta,J,U}[\tau_0\tau_x] = 0$).

2. Using the percolation representation of the AT model, we prove that this exponential decay extends to an open neigborhood of the self-dual line, and therefore $\beta_{\mathrm{sd}} < \beta_c^{\tau}$.

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2. Using the percolation representation of the AT model, we prove that this exponential decay extends to an open neigborhood of the self-dual line, and therefore $\beta_{\mathrm{sd}} < \beta_c^\tau$.
3. By duality relation in the percolation representation of the AT model, we prove that in an open neigborhood of the self-dual line, we have $\mathbb{E}_{\beta,J,U}[\tau_0\tau_0'\tau_x\tau_x'] > c$ for some $c > 0$ uniform in x. This proves that $\beta_{\mathrm{sd}} > \beta_c^{\tau\tau'}$.

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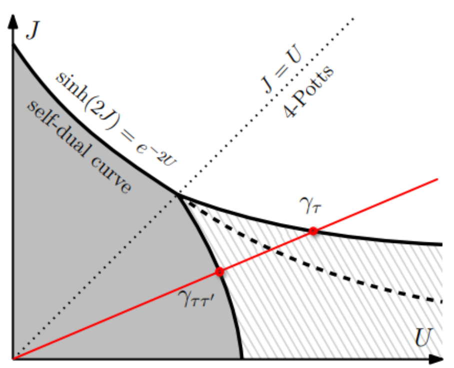

Further directions

1. Can we take J or U negative ?
2. Can we take the non-symmetrical model where the two copies of the Ising model do not have the same strength of the interaction and establish the complete phase diagram ?
3. Can we prove verify other physics' prediction ? (for instance, the continuity of the phase transition for $U > J$)

Thank you for your attention and let’s eat ! =)