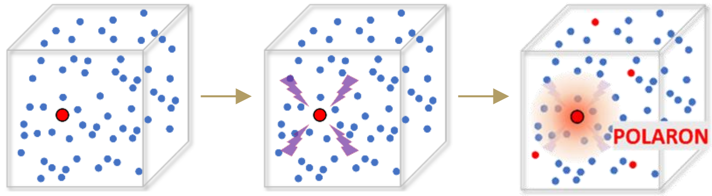
Quantissima in the Serenissima V, August 13, 2024

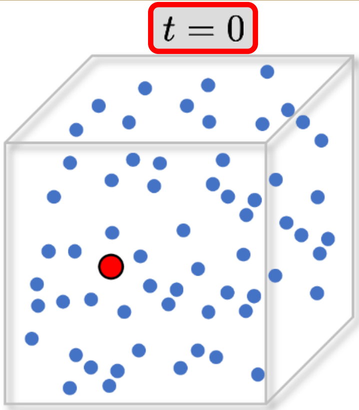
Effective Polaron Dynamics for an Impurity Particle Interacting with a Fermi gas

Duc Viet Hoang

based on joint work with Peter Pickl [arXiv:2406.08028]

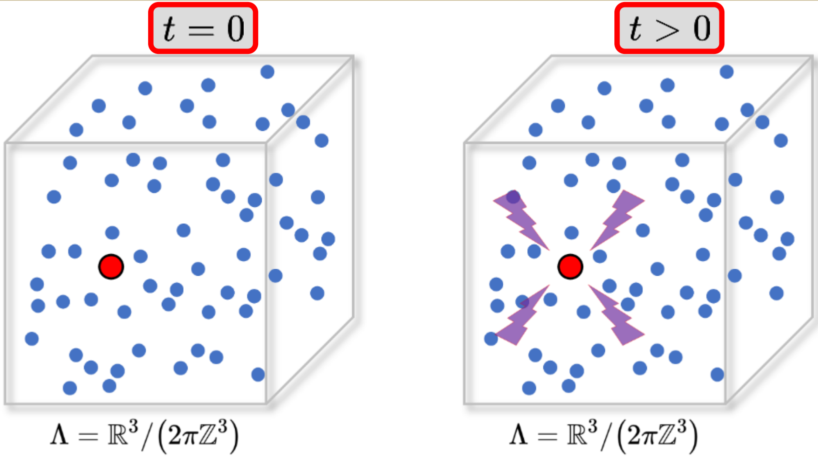


Setting & goal



$\Lambda = \mathbb{R}^3 / (2\pi\mathbb{Z}^3)$

Setting & goal

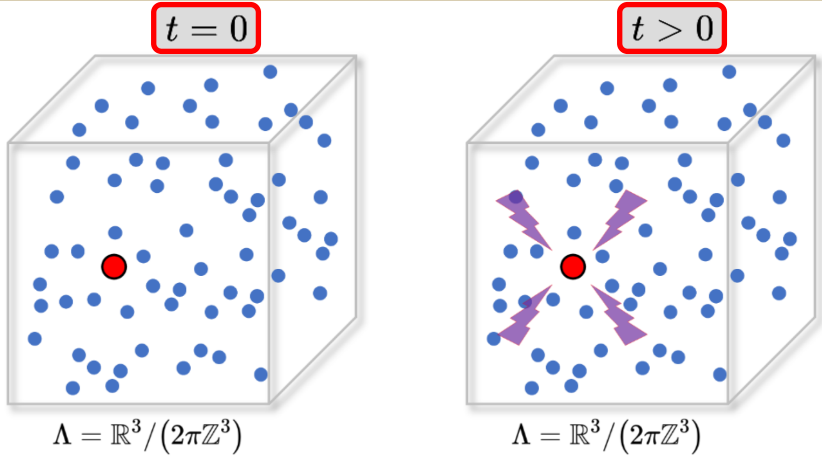


Setting & goal

Dense ideal gas of N (spin-less) fermions **interacting** with an impurity

$$H = -\Delta_y + \sum_{i=1}^{N} (-\Delta_{x_i}) + \sum_{i=1}^{N} V(x_i - y)$$

$\hat{V} \in C_c\left(\mathbb{Z}^3, \mathbb{R}_{\geq 0}\right)$
spherically sym.



Setting & goal

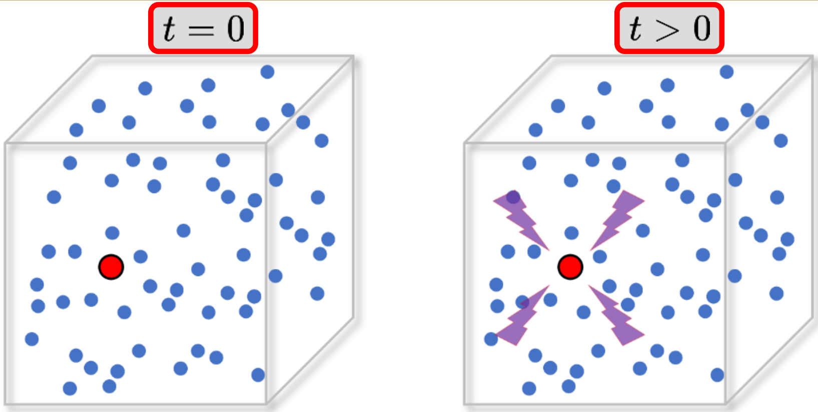
Dense ideal gas of N (spin-less) fermions **interacting** with an impurity

$$H = -\Delta_y + \sum_{i=1}^{N}(-\Delta_{x_i}) + \sum_{i=1}^{N} V(x_i - y)$$

$\hat{V} \in C_c(\mathbb{Z}^3, \mathbb{R}_{\geq 0})$
spherically sym.

$$i\partial_t \psi_t = H\psi_t, \qquad \psi_0 = \xi_0 \otimes \Omega_N$$

$$\in L^2(\Lambda, \mathrm{d}y) \otimes L^2(\Lambda)^{\wedge N}$$



$\Lambda = \mathbb{R}^3/(2\pi\mathbb{Z}^3)$ $\Lambda = \mathbb{R}^3/(2\pi\mathbb{Z}^3)$

Setting & goal

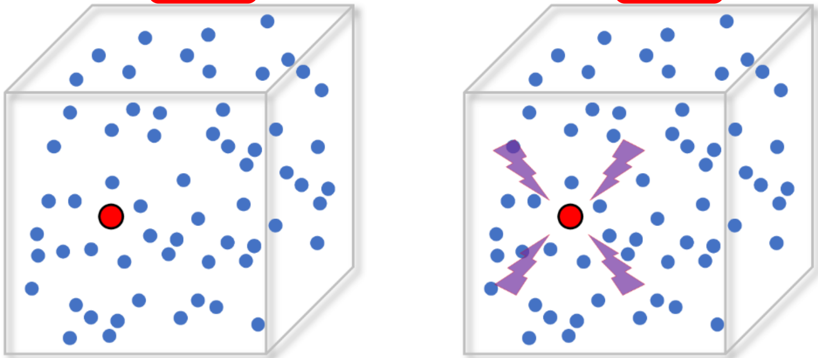
Dense ideal gas of N (spin-less) fermions **interacting** with an impurity

$$H = -\Delta_y + \sum_{i=1}^{N} (-\Delta_{x_i}) + \sum_{i=1}^{N} V(x_i - y)$$

$\hat{V} \in C_c(\mathbb{Z}^3, \mathbb{R}_{\geq 0})$
spherically sym.

$$i\partial_t \psi_t = H\psi_t, \qquad \psi_0 = \xi_0 \otimes \Omega_N \in L^2(\Lambda, \mathrm{d}y) \otimes L^2(\Lambda)^{\wedge N}$$

$t = 0$ $\qquad$ $t > 0$

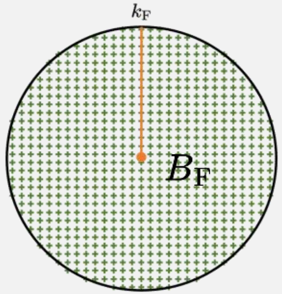


$\Lambda = \mathbb{R}^3/(2\pi\mathbb{Z}^3)$ $\qquad$ $\Lambda = \mathbb{R}^3/(2\pi\mathbb{Z}^3)$

$$\Omega_N = \bigwedge_{k \in B_{\mathrm{F}}} f_k$$

$N \equiv |B_{\mathrm{F}}| \sim k_{\mathrm{F}}^3$ with

$$B_{\mathrm{F}} = \{k \in \mathbb{Z}^3 : |k| \leq k_{\mathrm{F}}\}$$

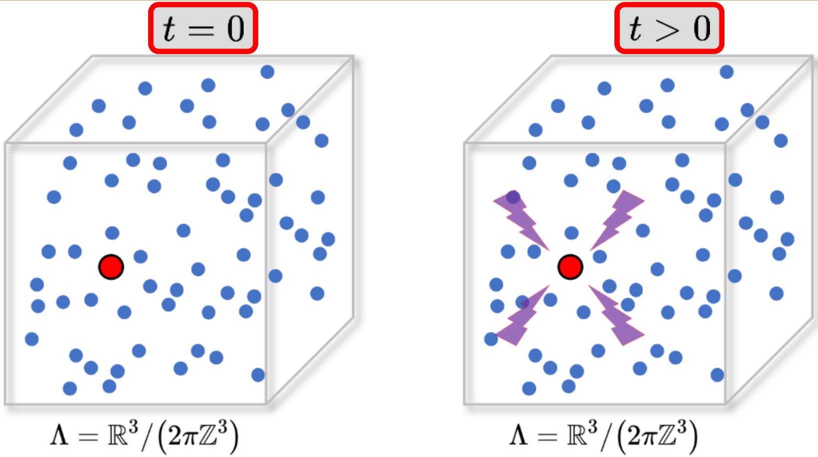


Setting & goal

Dense ideal gas of N (spin-less) fermions **interacting** with an impurity

$$H = -\Delta_y + \sum_{i=1}^{N} (-\Delta_{x_i}) + \sum_{i=1}^{N} V(x_i - y)$$

$\hat{V} \in C_c\left(\mathbb{Z}^3, \mathbb{R}_{\geq 0}\right)$
spherically sym.

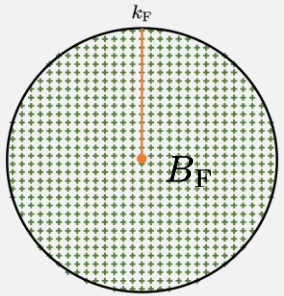


$$i\partial_t \psi_t = H\psi_t, \qquad \psi_0 = \xi_0 \otimes \Omega_N$$
$$\in L^2(\Lambda, \mathrm{d}y) \otimes L^2(\Lambda)^{\wedge N}$$

$$\Omega_N = \bigwedge_{k \in B_\mathrm{F}} f_k$$

$$N \equiv |B_\mathrm{F}| \sim k_\mathrm{F}^3 \quad \text{with}$$

$$B_\mathrm{F} = \{k \in \mathbb{Z}^3 : |k| \leq k_\mathrm{F}\}$$



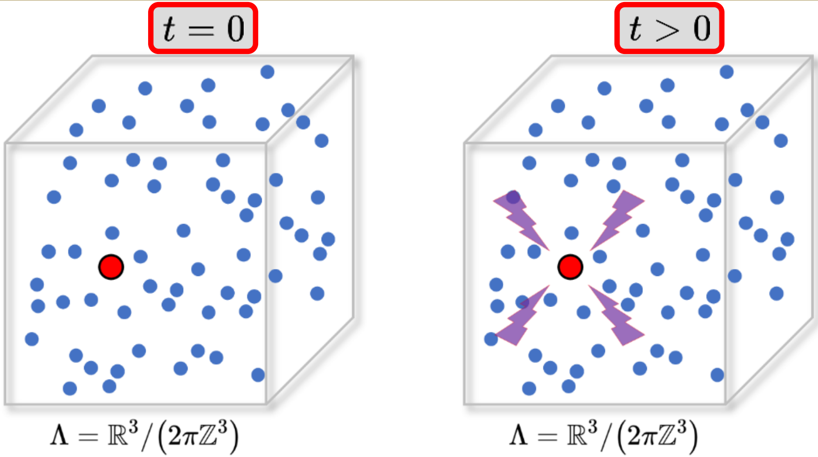
➢ **Goal:** $\psi_t \approx \psi_t^{\mathrm{eff}}$ for k_F large

Setting & goal

Dense ideal gas of N (spin-less) fermions **interacting** with an impurity

$$H = -\Delta_y + \sum_{i=1}^{N}(-\Delta_{x_i}) + \sum_{i=1}^{N} V(x_i - y)$$

$\hat{V} \in C_c\left(\mathbb{Z}^3, \mathbb{R}_{\geq 0}\right)$
spherically sym.



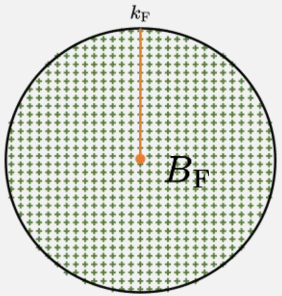
$$i\partial_t \psi_t = H\psi_t, \qquad \psi_0 = \xi_0 \otimes \Omega_N$$

$$\in L^2(\Lambda, \mathrm{d}y) \otimes L^2(\Lambda)^{\wedge N}$$

$$\Omega_N = \bigwedge_{k \in B_\mathrm{F}} f_k$$

$N \equiv |B_\mathrm{F}| \sim k_\mathrm{F}^3$ with

$$B_\mathrm{F} = \{k \in \mathbb{Z}^3 : |k| \leq k_\mathrm{F}\}$$

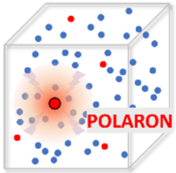


➤ **Goal:** $\psi_t \approx \psi_t^{\mathrm{eff}}$ for k_F large

i.e. $\|\psi_t - \psi_t^{\mathrm{eff}}\| \xrightarrow{k_\mathrm{F} \to \infty} 0$

The effective Polaron Hamiltonian

EBERHARD KARLS UNIVERSITÄT TÜBINGEN



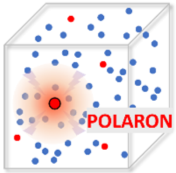
Theorem. [H, Pickl '24]

It holds for regular ξ_0

$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_{\mathrm{F}}^{-1}\right) : \qquad \left\|\psi_t - R e^{-i\mathbb{H}^{\mathrm{eff}} t}\, \xi_0 \otimes \Omega\right\| \xrightarrow{k_{\mathrm{F}} \to \infty} 0$$

The effective Polaron Hamiltonian

EBERHARD KARLS UNIVERSITÄT TÜBINGEN



Theorem. [H, Pickl '24]

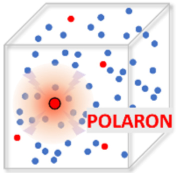
It holds for regular ξ_0

time scale of fastest fermions

$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_{\mathrm{F}}^{-1}\right) : \qquad \left\| \psi_t - R e^{-i\mathbb{H}^{\mathrm{eff}} t}\, \xi_0 \otimes \Omega \right\| \xrightarrow{k_{\mathrm{F}} \to \infty} 0$$

The effective Polaron Hamiltonian

EBERHARD KARLS UNIVERSITÄT TÜBINGEN



Theorem. [H, Pickl '24]

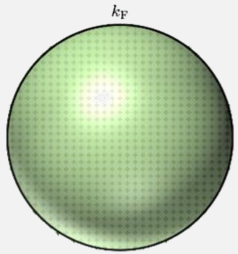
It holds for regular ξ_0

time scale of fastest fermions

$$\forall t \in [0, T] \text{ with } T \in \mathcal{O}\left(k_{\mathrm{F}}^{-1}\right): \qquad \|\psi_t - R e^{-i\mathbb{H}^{\mathrm{eff}} t}\, \xi_0 \otimes \Omega\| \overset{k_{\mathrm{F}} \to \infty}{\longrightarrow} 0$$

Unitary map $R : \mathcal{F} \to \mathcal{F}$

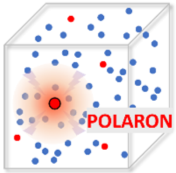
- tracks excitations **w.r.t. filled Fermi ball**
- filled Fermi ball $\longmapsto$ vacuum



$$\overset{R}{\longmapsto} \quad \Omega$$

PARTICLE-HOLE TF

The effective Polaron Hamiltonian



Theorem. [H, Pickl '24]

It holds for regular ξ_0

time scale of fastest fermions

$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_{\mathrm{F}}^{-1}\right): \qquad \|\psi_t - R e^{-i\mathbb{H}^{\mathrm{eff}} t}\, \xi_0 \otimes \Omega\| \overset{k_{\mathrm{F}} \to \infty}{\longrightarrow} 0$$

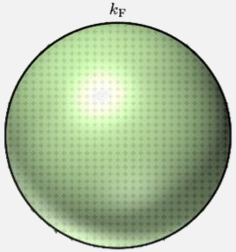
Fröhlich-type/Polaron Hamiltonian

$$\mathbb{H}^{\mathrm{eff}} = -\Delta_y + c^* c(\epsilon) + c^*(\hat{W}_y) + c(\hat{W}_y) + E_N^{\mathrm{pw}}$$

$$\text{with} \qquad \epsilon_\alpha(k) = 2k_F |k \cdot \hat{\omega}_\alpha|,$$
$$(\hat{W}_y)_\alpha(k) = n_\alpha(k) \hat{V}(k) e^{iky}$$

Unitary map $R : \mathcal{F} \to \mathcal{F}$

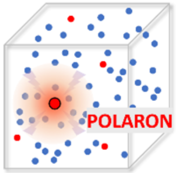
- tracks excitations **w.r.t. filled Fermi ball**
- filled Fermi ball $\longmapsto$ vacuum



$\overset{R}{\longmapsto} \quad \Omega$

PARTICLE-HOLE TF

The effective Polaron Hamiltonian



Theorem. [H, Pickl '24]

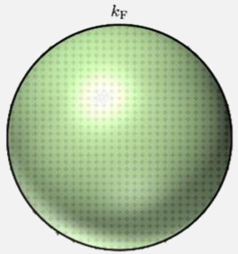
It holds for regular ξ_0

time scale of fastest fermions

$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_{\mathrm{F}}^{-1}\right): \qquad \|\psi_t - Re^{-i\mathbb{H}^{\text{eff}}t}\,\xi_0 \otimes \Omega\| \overset{k_{\mathrm{F}}\to\infty}{\longrightarrow} 0$$

Unitary map $R : \mathcal{F} \to \mathcal{F}$

- tracks excitations **w.r.t. filled Fermi ball**
- filled Fermi ball $\longmapsto$ vacuum



$$\overset{R}{\longmapsto} \quad \Omega$$

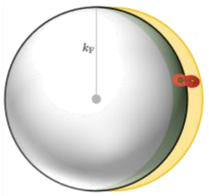
PARTICLE-HOLE TF

Fröhlich-type/Polaron Hamiltonian

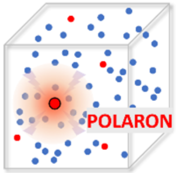
$$\mathbb{H}^{\text{eff}} = -\Delta_y + c^*c(\epsilon) + c^*(\hat{W}_y) + c(\hat{W}_y) + E_N^{\text{pw}}$$

c^*, c - operators:

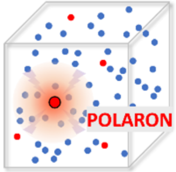
- **almost-bosonic** operators **constructed** from fermionic operators
- associated to **delocalized particle-hole excitations**



The effective coupled coherent state



The effective coupled coherent state



Theorem. [H, Pickl '24]

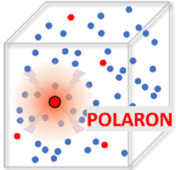
There exists a $\eta_t^y \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ s.t. for regular ξ_0

$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_{\mathrm{F}}^{-1}\right): \quad \|\psi_t - e^{iP(t)} R\, W(\eta_t^y)\xi_0 \otimes \Omega\| \overset{k_{\mathrm{F}}\to\infty}{\longrightarrow} 0$$

Almost-bosonic Weyl operator

$$W(\eta) = e^{c^*(\eta)-c(\eta)}$$

The effective coupled coherent state



Theorem. [H, Pickl '24]

There exists a $\eta_t^y \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ s.t. for regular ξ_0

$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_{\mathrm{F}}^{-1}\right): \qquad \|\psi_t - e^{iP(t)} R\, W(\eta_t^y)\xi_0 \otimes \Omega\| \xrightarrow{k_{\mathrm{F}}\to\infty} 0$$

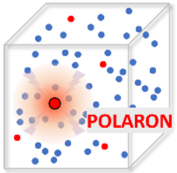
Almost-bosonic Weyl operator

$$W(\eta) = e^{c^*(\eta)-c(\eta)}$$

- **Coupled coherent state** as Poisson distribution in Fock space

$$e^{-\|\eta\|^2/2}\left\{\xi_0,\ \eta\xi_0,\ \frac{\eta^{\otimes 2}\xi_0}{\sqrt{2!}}, \dots, \frac{\eta^{\otimes n}\xi_0}{\sqrt{n!}}, \dots\right\}$$

The effective coupled coherent state



Theorem. [H, Pickl '24]

There exists a $\eta_t^y \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ s.t. for regular ξ_0

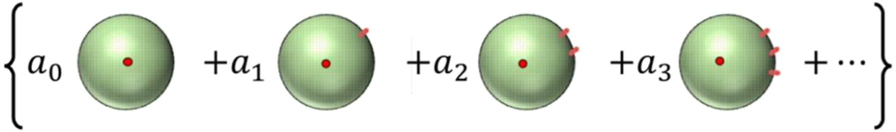
$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_F^{-1}\right): \quad \|\psi_t - e^{iP(t)} R\, W(\eta_t^y)\xi_0 \otimes \Omega\| \overset{k_F\to\infty}{\longrightarrow} 0$$

Almost-bosonic Weyl operator

$$W(\eta) = e^{c^*(\eta) - c(\eta)}$$

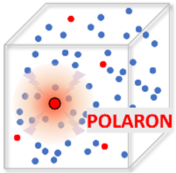
- **Coupled coherent state** as Poisson distribution in Fock space

$$e^{-\|\eta\|^2/2}\left\{\xi_0,\ \eta\xi_0,\ \frac{\eta^{\otimes 2}\xi_0}{\sqrt{2!}}, \dots, \frac{\eta^{\otimes n}\xi_0}{\sqrt{n!}}, \dots\right\}$$



The effective coupled coherent state

EBERHARD KARLS UNIVERSITÄT TÜBINGEN



Theorem. [H, Pickl ’24]

There exists a $\eta_t^y \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ s.t. for regular ξ_0

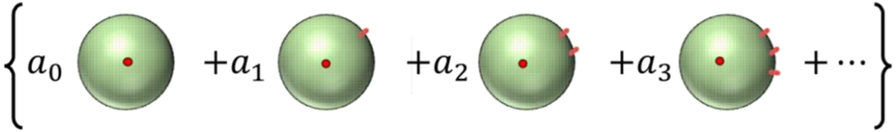
$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_F^{-1}\right): \quad \| \psi_t - e^{iP(t)} R\, W(\eta_t^y)\xi_0 \otimes \Omega \| \xrightarrow{k_F \to \infty} 0$$

Almost-bosonic Weyl operator

$$W(\eta) = e^{c^*(\eta) - c(\eta)}$$

➢ **Coupled coherent state** as Poisson distribution in Fock space

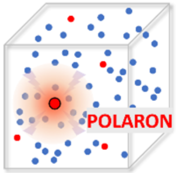
$$e^{-\|\eta\|^2/2} \left\{ \xi_0,\ \eta\xi_0,\ \frac{\eta^{\otimes 2}\xi_0}{\sqrt{2!}}, \dots, \frac{\eta^{\otimes n}\xi_0}{\sqrt{n!}}, \dots \right\}$$



Mean number of excitations

$$\langle \mathcal{N} \rangle_{W(\eta_t^y)\xi_0 \otimes \Omega} \simeq 2\|\eta_t^y\|^2,$$

The effective coupled coherent state



Theorem. [H, Pickl '24]

There exists a $\eta_t^y \in \bigoplus_{k\in\Gamma} l^2(\mathcal{I}_k)$ s.t. for regular ξ_0

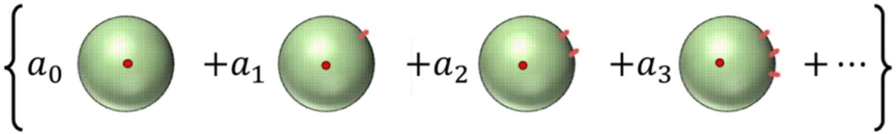
$$\forall t \in [0,T] \text{ with } T \in \mathcal{O}\left(k_F^{-1}\right): \quad \|\psi_t - e^{iP(t)} R\, W(\eta_t^y)\xi_0 \otimes \Omega\| \xrightarrow{k_F \to \infty} 0$$

Almost-bosonic Weyl operator

$$W(\eta) = e^{c^*(\eta) - c(\eta)}$$

➢ **Coupled coherent state** as Poisson distribution in Fock space

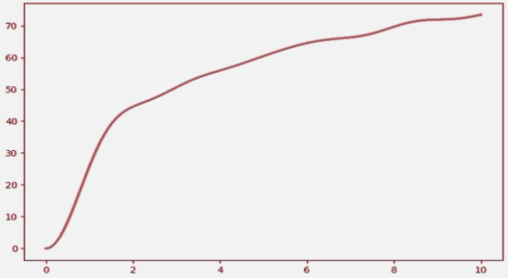
$$e^{-\|\eta\|^2/2}\left\{\xi_0,\ \eta\xi_0,\ \frac{\eta^{\otimes 2}\xi_0}{\sqrt{2!}}, \dots, \frac{\eta^{\otimes n}\xi_0}{\sqrt{n!}}, \dots\right\}$$



Mean number of excitations

$$\langle \mathcal{N} \rangle_{W(\eta_t^y)\xi_0 \otimes \Omega} \simeq 2\|\eta_t^y\|^2,$$

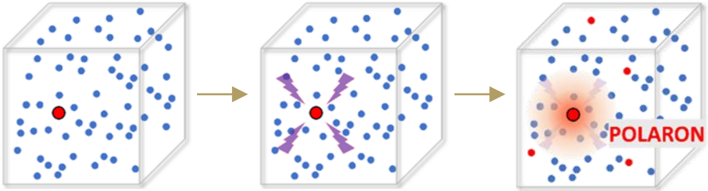
$$\|\eta_t\|^2 \simeq \pi \sum_{k\in\mathbb{Z}^3} \frac{\hat{V}(k)^2}{|k|}\{\log(2k_F|k|t) - \mathrm{Ci}(2k_F|k|t) + \gamma\}$$



t in k_F^{-1}

Summary & outlook

- Approximation for large k_F : $\|\psi_t - \psi_t^{\mathrm{eff}}\| \xrightarrow{k_F \to \infty} 0$
  - on level of Hamiltonian: $\psi_t^{\mathrm{eff},1} = e^{-iPt} R e^{-iH^{\mathrm{eff}}t}$ *Fröhlich-type Hamiltonian*
  - on level of states: $\psi_t^{\mathrm{eff},2} = e^{-iP(t)} RW(\eta_t)\psi_0$ *coupled coherent state*
- effective number of excitations



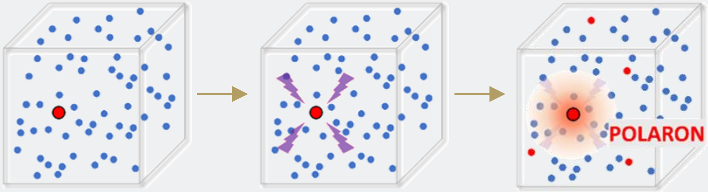
Summary & outlook

- ➢ Approximation for large k_F : $\|\psi_t - \psi_t^{\text{eff}}\| \xrightarrow{k_F\to\infty} 0$
  - on level of Hamiltonian: $\psi_t^{\text{eff},1} = e^{-iPt} R e^{-iH^{\text{eff}}t}$ *Fröhlich-type Hamiltonian*
  - on level of states: $\psi_t^{\text{eff},2} = e^{-iP(t)} RW(\eta_t)\psi_0$ *coupled coherent state*
- ➢ effective number of excitations

Open questions & further direction

1. Long-time behavior? (polaron has finite lifetime)
2. Effective decoupling $W(\eta_t)\xi_0 \otimes \Omega$ by seperation of time-scales?

Generalize to
- wider class of (internal) interactions
- more impurities
- trap potential, TD limit



Thank you for your attention!