Universal behaviour of the BCS energy gap

Asbjørn Bækgaard Lauritsen
alaurits@ist.ac.at

Institute of Science and Technology Austria

Joint with Joscha Henheik

arXiv:2312.11310

Venice
2024-08-13

Bardeen–Cooper–Schrieffer (BCS) 1957 theory of superconductivity

Described by **BCS gap equation**

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{V}(p-q) \frac{\Delta(q)}{K_T^{\Delta}(q)} \, \mathrm{d}q,$$

with

$$K_T^{\Delta}(p) = \frac{E_{\Delta}(p)}{\tanh\left(\frac{E_{\Delta}(p)}{2T}\right)}, \qquad E_{\Delta}(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}.$$

Electron-lattice interaction leads to effective attractive electron-electron interaction V.

Material is superconducting if non-zero solution Δ exists.

Bardeen–Cooper–Schrieffer (BCS) 1957 theory of superconductivity

Described by **BCS gap equation**

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{V}(p-q) \frac{\Delta(q)}{K_T^\Delta(q)} \, \mathrm{d}q,$$

with

$$K_T^\Delta(p) = \frac{E_\Delta(p)}{\tanh\left(\frac{E_\Delta(p)}{2T}\right)}, \qquad E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}.$$

Electron-lattice interaction leads to effective attractive electron-electron interaction V.

Material is superconducting if non-zero solution Δ exists.

$\rightsquigarrow$ Define the **critical temperature** $T_c := \inf\{T \geq 0 : \text{no non-zero solution}\}$.

Phase transition at T_c: superconducting for $T < T_c$ and normal metal for $T \geq T_c$.

$E_\Delta(p)$ has interpretation of dispersion relation for associated quasi-particles

$\rightsquigarrow$ Define the **energy gap** $\Xi = \Xi(T) := \inf E_\Delta(p)$

Bardeen–Cooper–Schrieffer (BCS) 1957 theory of superconductivity

Described by **BCS gap equation**

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{V}(p-q) \frac{\Delta(q)}{K_T^\Delta(q)} \, \mathrm{d}q,$$

with

$$K_T^\Delta(p) = \frac{E_\Delta(p)}{\tanh\left(\frac{E_\Delta(p)}{2T}\right)}, \qquad E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}.$$

Electron-lattice interaction leads to effective attractive electron-electron interaction V.

Material is superconducting if non-zero solution Δ exists.

$\rightsquigarrow$ Define the **critical temperature** $T_c := \inf\{T \geq 0 : \text{no non-zero solution}\}$.

Phase transition at T_c: superconducting for $T < T_c$ and normal metal for $T \geq T_c$.

$E_\Delta(p)$ has interpretation of dispersion relation for associated quasi-particles

$\rightsquigarrow$ Define the **energy gap** $\Xi = \Xi(T) := \inf E_\Delta(p)$

T_c and Ξ are material-dependent (μ, V-dependent). Both measure "stability" of superconducting phase.

Question

Relation between T_c and Ξ ?

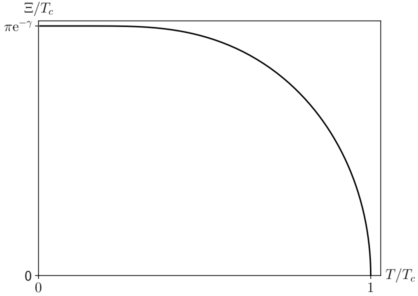
Physics prediction for T_c small:

$$\frac{\Xi}{T_c} \approx f_{\mathrm{BCS}}\left(\sqrt{1-\frac{T}{T_c}}\right)$$

for **universal** function f_{BCS}.
For $T = 0$ and $T \to T_c$:

$$\frac{\Xi(T=0)}{T_c} \approx \pi e^{-\gamma} \approx 1.76,$$

$$\frac{\Xi}{T_c} \approx \sqrt{\frac{8\pi^2}{7\zeta(3)}}\sqrt{1-\frac{T}{T_c}} \approx 3.06\sqrt{1-\frac{T}{T_c}}.$$



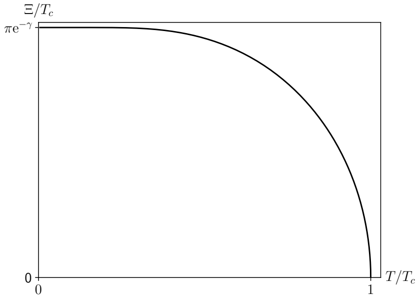
Physics prediction for T_c small:

$$\frac{\Xi}{T_c} \approx f_{\mathrm{BCS}}\left(\sqrt{1-\frac{T}{T_c}}\right)$$

for **universal** function f_{BCS}.
For $T=0$ and $T \to T_c$:

$$\frac{\Xi(T=0)}{T_c} \approx \pi e^{-\gamma} \approx 1.76,$$

$$\frac{\Xi}{T_c} \approx \sqrt{\frac{8\pi^2}{7\zeta(3)}}\sqrt{1-\frac{T}{T_c}} \approx 3.06\sqrt{1-\frac{T}{T_c}}.$$



Theorem (Henheik–L. 2023)

Replace V by λV. Then, in weak coupling limit $\lambda \to 0$ picture holds.

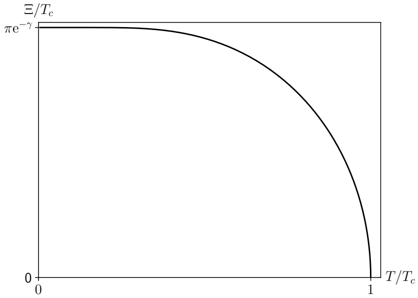
Physics prediction for T_c small:

$$\frac{\Xi}{T_c} \approx \mathrm{f}_{\mathrm{BCS}}\left(\sqrt{1-\frac{T}{T_c}}\right)$$

for **universal** function $\mathrm{f}_{\mathrm{BCS}}$.
For $T = 0$ and $T \to T_c$:

$$\frac{\Xi(T=0)}{T_c} \approx \pi e^{-\gamma} \approx 1.76,$$

$$\frac{\Xi}{T_c} \approx \sqrt{\frac{8\pi^2}{7\zeta(3)}}\sqrt{1-\frac{T}{T_c}} \approx 3.06\sqrt{1-\frac{T}{T_c}}.$$



Theorem (Henheik–L. 2023)

Replace V by λV. Then, in weak coupling limit $\lambda \to 0$ picture holds.

More precisely let $h = \sqrt{1 - T/T_c}$ then for some $c, c' > 0$

$$\frac{\Xi}{T_c} = \mathrm{f}_{\mathrm{BCS}}(h)\left(1 + \min\left\{O(h^{-1}e^{-c/\lambda}),\ O(he^{c'/\lambda}) + o_{\lambda\to 0}(1)\right\}\right)$$

with $o_{\lambda\to 0}(1)$ vanishing as $\lambda \to 0$ uniformly in h.

Main ideas of proof

Δ solves BCS gap equation

$\Updownarrow$

$K_T^\Delta + V$ has lowest eigenvalue 0 and associated eigenvector $\alpha = -\frac{\Delta}{2K_T^\Delta}$

$\Updownarrow$ Birman–Schwinger principle

Birman–Schwinger operator $B_{T,\Delta} = \operatorname{sgn} V|V|^{1/2}(K_T^\Delta)^{-1}|V|^{1/2}$ has -1 as lowest eigenvalue and associated eigenvector $\phi = \operatorname{sgn} V|V|^{1/2}\alpha$.

Proof goes as follows:

- Decompose $B_{T,\Delta}$ into dominant term of order $m(T,\Delta) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{|p|\le\sqrt{2\mu}} \frac{1}{K_T^\Delta(p)}\,\mathrm{d}p \sim \log \mu/(T+\Delta)$ and error term of order 1.
- Study spectral properties of $B_{T,\Delta}$ for T and for $T = T_c$ to get formula $m(T,\Delta) - m(T_c, 0) = O(e^{-c/\lambda})$
- Observe that $m(T,\Delta) - m(T_c, 0) = O(e^{-c/\lambda})$ is (almost) defining equation for f_{BCS}. $\square$