On Fermi Gases and their Momentum Distribution

Joint works with Niels Benedikter, Emanuela Giacomelli, Asbjørn Bækgaard Lauritsen and Diwakar Naidu

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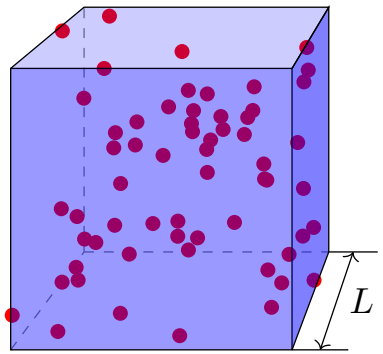
Model

- N fermions on a torus $[0, L]^3$
  $\Rightarrow$ the density is $\rho = \frac{N}{L^3}$
- Hilbert space: $\mathcal{H}^{(N)} := L^2([0, L]^3)^{\otimes_a N}$
  That means, $\psi \in \mathcal{H}^{(N)}$ is antisymmetric
  $\psi(\ldots, x_i, \ldots, x_j, \ldots) = -\psi(\ldots, x_j, \ldots, x_i, \ldots)$.

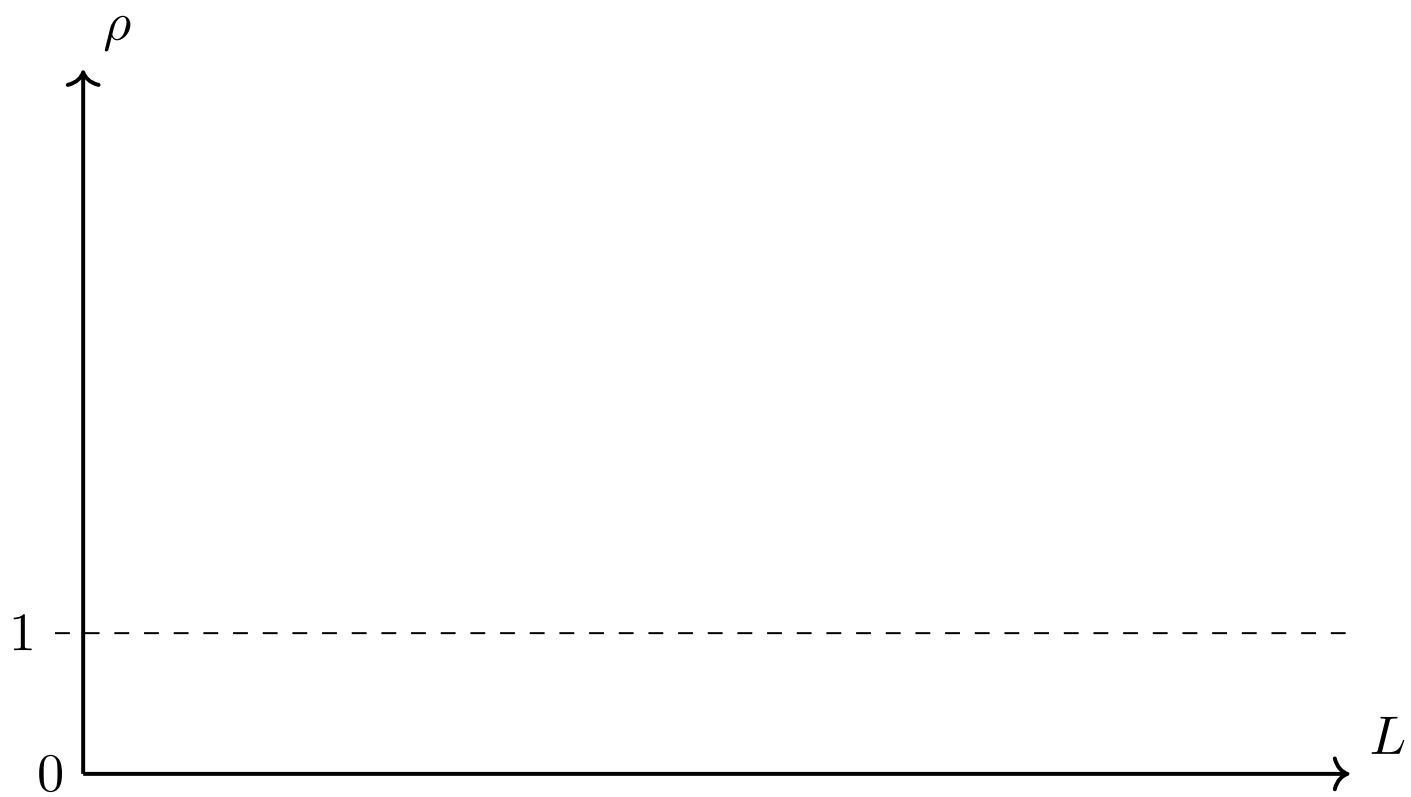


- Hamiltonian $H_N : \mathcal{H}^{(N)} \supset \mathrm{dom}(H_N) \to \mathcal{H}^{(N)}$,

$$H_N := \rho^{-2/3} \sum_{j=1}^{N} -\Delta_{x_j} + \rho^{-1} \sum_{i<j}^{N} V(x_i - x_j)$$

- Applications: describing electrons in metals, nuclear matter, computational quantum chemistry, ultra-cold gases

- (ρ, L) are the parameters
- Physically, we want to take $N = \rho L^3 \to \infty$.



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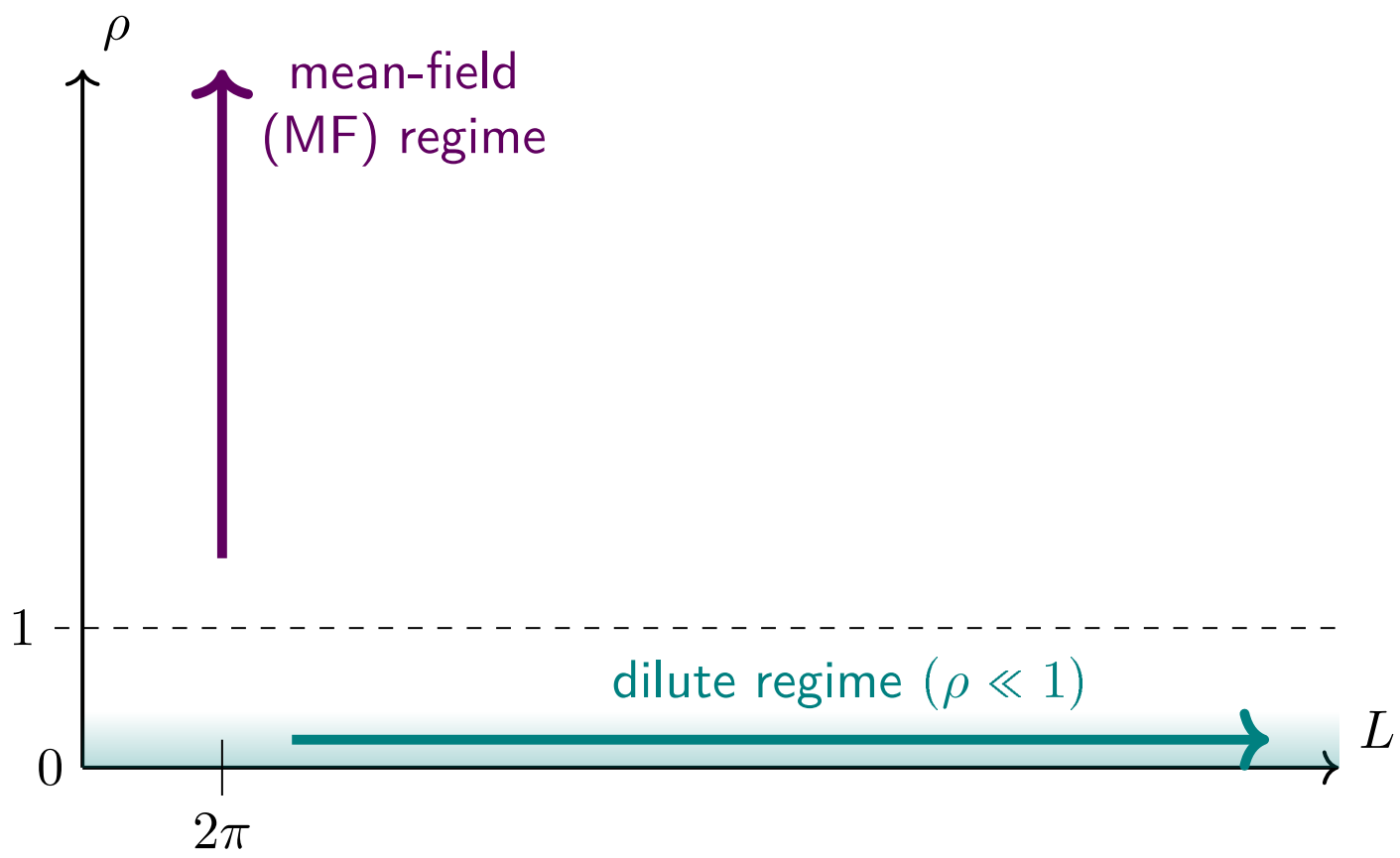

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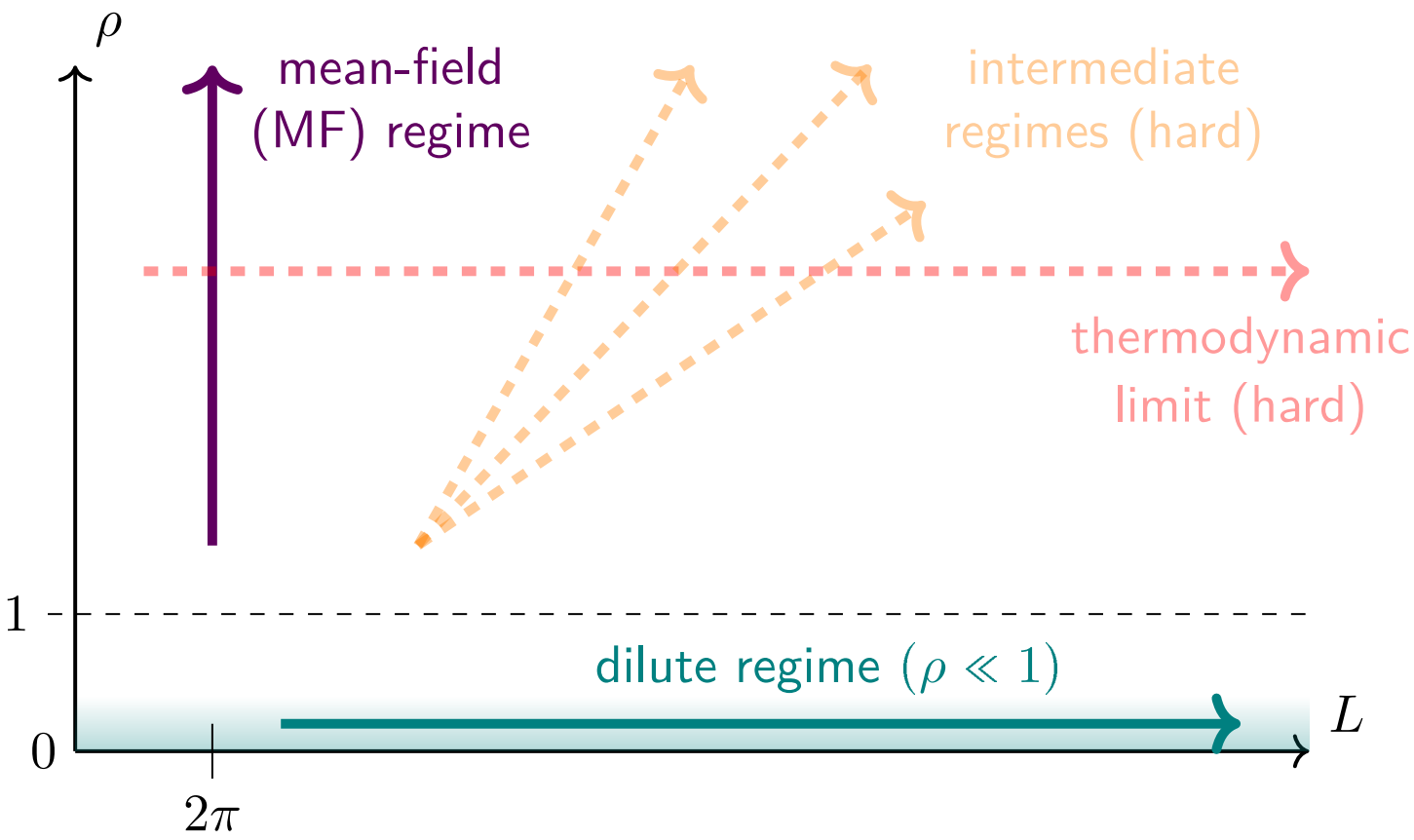
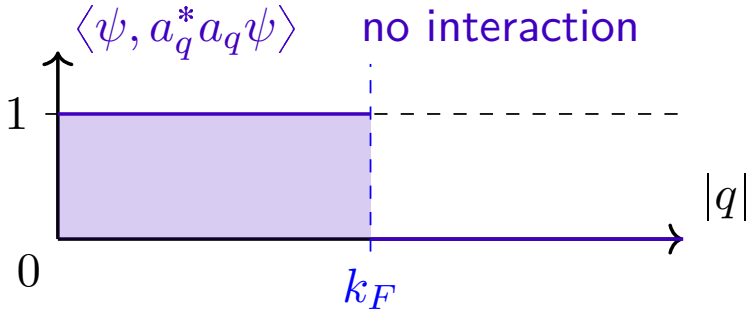
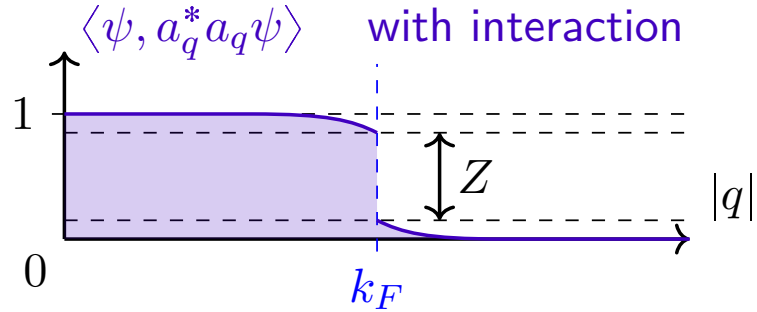


Interesting Quantities

- **Ground state energy** $E_{\mathrm{GS}} := \inf \sigma(H_N)$, conjectured in 50's.
- **Momentum distribution** $\boxed{n_q := \langle \psi, a_q^* a_q \psi \rangle}$ for $q \in \mathbb{R}^3, \psi \approx \psi_{\mathrm{GS}}$, conjectured in 60's.





- A jump with height Z is expected in $d = 3$ dimensions.
- $Z > 0$ indicates the presence of "quasiparticles", which are essential in **Fermi liquid theory** [Landau 1956-1959]

Existing Results

- Can Fermi gases be described as Fermi liquids?
- $d = 1$: No, we rather have a Luttinger liquid.
- $d = 2$: Proofs for Fermi liquid exist, using mathematical Renormalization Group (RG) techniques.
- $d = 3$: **Open question**.

Theorem (Momentum distribution, [Benedikter, L. 2023])

For $\hat{V} \geqslant 0$ compactly supported, $\exists$ "RPA trial states" (ψ_N) with $\langle \psi_N, H_N \psi_N \rangle - E_{\mathrm{GS}} = \mathcal{O}(N^{-1/3-\alpha})$, s.t. for most $q \in \mathbb{Z}^3$,

$$n_q = \begin{cases} 1 - N^{-\frac{2}{3}} I(q) + \mathcal{O}(N^{-\frac{2}{3}-\frac{1}{12}}) & \text{for } |q| < k_{\mathrm{F}} \\ N^{-\frac{2}{3}} I(q) + \mathcal{O}(N^{-\frac{2}{3}-\frac{1}{12}}) & \text{for } |q| \geqslant k_{\mathrm{F}} \end{cases},$$

with an explicit $I(q) \sim N^0$. Further, $Z \geqslant 1 - \mathcal{O}(N^{-\frac{2}{3}+\frac{1}{12}})$.

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Theorem (Momentum distribution, [Benedikter, L. 2023])

For $\hat{V} \geqslant 0$ *compactly supported,* $\exists$ *"RPA trial states"* (ψ_N) *with* $\langle \psi_N, H_N \psi_N \rangle - E_{\mathrm{GS}} = \mathcal{O}(N^{-1/3-\alpha})$*, s.t. for most* $q \in \mathbb{Z}^3$*,*

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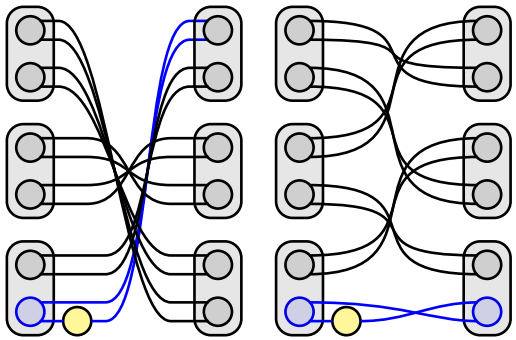
Existing Results

- [L. 2024] re-establishes the formula for n_q in the trial states ψ_N using **Friedrichs diagrams** [Brooks, L. 2023]



- Techniques of [Benedikter, L. 2023] have been used in recent studies on E_{GS} in the MF and dilute regime:
  - [Benedikter, Nam, Porta, Schlein, Seiringer 2020–24],
  - [Christiansen, Hainzl, Nam 2022–24],
  - [Fournais, Ruba, Solovej 2024],
  - [Falconi, Giacomelli, Hainzl, Porta 2020–24],
  - [Lauritsen, Seiringer 2023–24], ...

Open Research Questions

- Finding n_q for other ψ and more singular potentials V in MF regime: work with N. Benedikter and D. Naidu (U Milan)
- Finding n_q in the dilute regime: work with N. Benedikter, E. Giacomelli (LMU Munich) and A.B. Lauritsen (IST Austria)
- Finding n_q for the true ground state
- Justifying next terms in the formulas for n_q or E_{GS} for the MF or dilute regime
- Finding n_q or E_{GS} in the thermodynamic limit or intermediate regimes
- ...

Thank you for your attention!