Gibbs measures and mathematical physics

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Schrödinger Equation

- Consider the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t \varphi = -\Delta\varphi + |\varphi|^2\varphi \\ \varphi_0 \in H^s(\mathbb{T}^d), \end{cases}$$

  where we take $(x, t) \in \mathbb{T}^d \times \mathbb{R}$.
- $s \in \mathbb{R}$ is the **Sobolev regularity** ("large s is high regularity")
- We have two **conserved** quantities associated with the Hartree equation:

$$M(\varphi) := \int dx\, |\varphi(x)|^2,$$

$$H(\varphi) := \int dx\, |\nabla\varphi(x)|^2 + \int dx\, |\varphi(x)|^4$$

- Idea: Use **conservation of energy** to build global solutions from local solutions

Gibbs Measure I

- **Problem**: Conservation laws are only defined at high regularity.
- At **low regularity**, we need a "substitute for conservation laws."
- We define the **Gibbs measure**

$$d\mathbb{P}_{Gibbs} := \frac{1}{z} e^{-H(\varphi)} d\varphi,$$

  which is supported on the space of initial conditions. Here
  - z is the **partition function**
  - H is the **Hamiltonian**
  - $d\varphi$ is the (formal) **infinite dimensional Lebesgue measure**
- Formally, the Gibbs measure **invariant** under the flow of the PDE (**Liouville's theorem**) – substitute for the conservation of energy.

References: **Constructive field theory** – Nelson, Glimm-Jaffe, Simon.

Gibbs Measure II

- **Advantages**:
  - Functions in support of Gibbs measure given by a **random Fourier series** – probabilistic cancellation (**Wick's theorem**).
  - Can talk about **probabilistic well posedness**.
  - Gibbs measures is supported on **low Sobolev regularity** function
  - Can use **invariance** of Gibbs measure to build **global solutions**.
- **Difficulties:**
  - PDE is **infinite dimensional** – Lebesgue measure is ill-defined (use **Wiener/Gaussian measures**).
  - Proving **normalisability** – need to show that z is finite.
  - **Regularity Issues** – higher dimensional measures are supported on lower regularity functions (distributions for $d = 2, 3$) – require **renormalisation**.

Open problem: Construction of the Gibbs measure for nonlinear Schrödinger equation for $d = 3$.

(Non-exhaustive) References: **Dispersive PDEs** – Lebowitz-Rose-Speer, Zhidkov, Bourgain, Deng, Nahmod, Yue, Oh, Sosoe, Tolomeo, Bringmann, Dinh, Rougerie, Wang, . . . (many others)

Mathematical Physics

- **Microscopic derivations of Gibbs measures**: "What do Gibbs measures correspond to on the many-body side?"
  - **(Very) Formal answer**:

$$\lim_{\text{Number of particles}\to\infty}\left[\frac{1}{Z_N}e^{-\frac{1}{N}H_N}\right] \text{``}=\text{''}\ \mathbb{P}_{\text{Gibbs}}$$

  - Approaches: Gibbs variational principle, perturbation theory, functional integrals

References: Lewin-Nam-Rougerie, Fröhlich-Knowles-Schlein-Sohinger, R-Sohinger.