Lace expansion

for random walks and statistical mechanics models

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Literature

- **Brydges, Spencer** '85: random walks
- **Bolthausen, Van der Hofstad, Kozma** '16 – *Lace expansion for dummies*
- **Hara, Van der Hofstad, Slade** '03: percolation
- **Sakai** '07: Ising model
- **Avena, Bolthausen, Ritzmann** '16: local CLT for self-avoiding walks
- **Brydges, Helmuth, Holmes** '21: $g|\varphi|^4$ model

Ising model on $\Lambda \subset \mathbb{Z}^d$

$$\Lambda \subset \mathbb{Z}^d, \quad \varphi = (\varphi_x)_{x\in\Lambda} \in \{\pm 1\}^\Lambda, \quad J_{x,y} \in \mathbb{R},\ \beta \geq 0$$

$$H_\Lambda(\varphi) := -\sum_{\{x,y\}\subset\Lambda} J_{x,y}\varphi_x\varphi_y$$

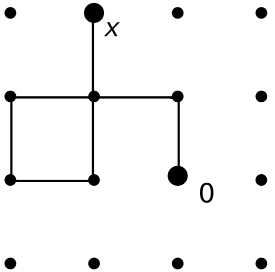
$$f\colon \{\pm 1\}^\Lambda \to \mathbb{R}, \quad \langle f\rangle_{\beta,\Lambda} := \frac{\sum_{\varphi\in\{\pm1\}^\Lambda} f(\varphi)\ \mathrm{e}^{-\beta H_\Lambda}}{\sum_{\varphi\in\{\pm1\}^\Lambda} \mathrm{e}^{-\beta H_\Lambda}}$$

Two-point correlation function, i.e. Green's function:

$$G_\beta(x) := \lim_{\Lambda\uparrow\mathbb{Z}^d} \langle \varphi_0\varphi_x\rangle_{\beta,\Lambda}$$

Green's function can be represented in terms of random walks.

Random walks



$\mathbb{Z}^d$, $\mathbf{d \geq 5}$

$$G_z(x) := \sum_{\gamma:\, 0 \to x} z^{|\gamma|} \prod_{0 \leq s < t \leq |\gamma|} \mathrm{e}^{-U_{st}(\gamma)}$$

$z \in (0, 1)$

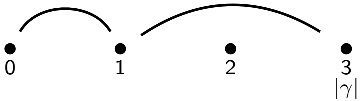
$$U_{st}(\gamma) = \begin{cases} 0, & \gamma(s) \neq \gamma(t) \\ \lambda > 0, & \gamma(s) = \gamma(t) \end{cases}$$

Critical z_c: as $|x| \to \infty$:

$$G_z(x) \sim \begin{cases} \mathrm{e}^{-m z |x|} & z < z_c \\ |x|^{2-d} & z = z_c \\ \text{diverges} & z > z_c \end{cases}$$

Ordered graphs

$$\prod_{0\le s<t\le|\gamma|} \mathrm{e}^{-U_{st}(\gamma)} = \prod_{0\le s<t\le|\gamma|} \left[1-\left(1-\mathrm{e}^{-U_{st}(\gamma)}\right)\right] = \sum_{\Gamma\in\text{graphs}} \prod_{st\in\Gamma} \left(1-\mathrm{e}^{-U_{st}(\gamma)}\right)$$



$$\left(1-\mathrm{e}^{-U_{0,1}(\gamma)}\right)\left(1-\mathrm{e}^{-U_{1,3}(\gamma)}\right)$$

$$G_z(x) = \sum_{\gamma:\,0\to x} z^{|\gamma|} \sum_{\Gamma\in\text{graphs}} \prod_{st\in\Gamma} \left(1-\mathrm{e}^{-U_{st}(\gamma)}\right)$$

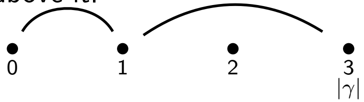
Replace graphs with connected graphs

$$G_z(x) = \sum_{\gamma\colon 0\to x} z^{|\gamma|} \sum_{\Gamma \in \text{graphs}} \prod_{st\in\Gamma} \left(1 - \mathrm{e}^{-U_{st}(\gamma)}\right)$$

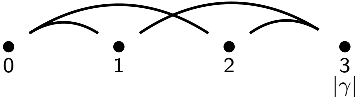
$$\Pi_z(x) := \sum_{\gamma\colon 0\to x} z^{|\gamma|} \sum_{\Gamma \in \substack{\text{connected}\\ \text{graphs}}} \prod_{st\in\Gamma} \left(1 - \mathrm{e}^{-U_{st}(\gamma)}\right)$$

$$G_z = S_z + S_z * \Pi_z * G_z, \quad S_z(x) := \sum_{\gamma\colon 0\to x} z^{|\gamma|}$$

Connected: 0 and $|\gamma|$ belong to an edge. Every other vertex has an edge above it.

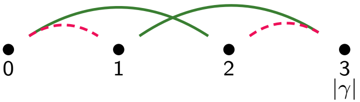


Not connected

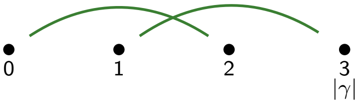


Connected

$$\Pi_z(x) = \sum_{\gamma\colon 0 \to x} z^{|\gamma|} \sum_{\text{laces}} \sum_{\substack{\text{additional} \\ \text{edges}}} \prod_{st \in \Gamma} \left(1 - \mathrm{e}^{-U_{st}(\gamma)}\right)$$



Not minimally connected



Minimally connected = lace

Applications

The lace expansion can be used to:

- Get asymptotics for a wider range of parameters,
- Remove some assumptions.

We hope to use it on more models.

Thank you!