Derivation of the effective dynamics for the Bose Polaron

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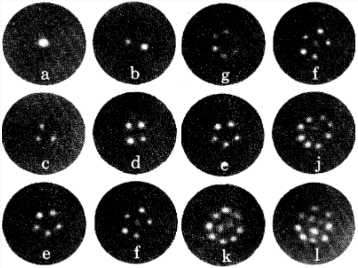
Quantum gas of N bosons with 1 impurity particle $\rightarrow$ Tracer particle.

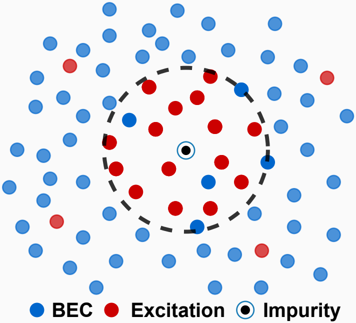
Applications of **impurity** systems into physics:

- **Track** local structure:
  Vortex lattice in a boson gas.
- Local **manipulation**:
  Sympathetic cooling.



Motivation

Quantum gas of N bosons with 1 impurity particle $\rightarrow$ Tracer particle.



Applications of **impurity** systems into physics:

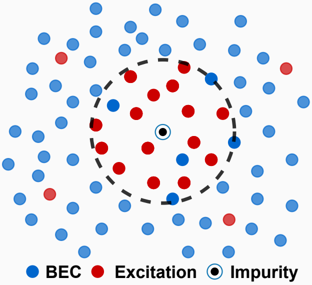
- **Track** local structure:
  Vortex lattice in liquid helium.
- Local **manipulation**:
  Sympathetic cooling.

Bose Polaron: Quasi-particle of impurity and bosons.

Goal: Prove the existence of a Bose Polaron in our system.
$\rightarrow$ Effective dynamics.

- Quantum gas of N bosons, 1 impurity particle in $\mathbb{R}^3$.
- **Volume** Λ, **density** $\rho = \frac{N}{\Lambda}$ with ρ, Λ **large**.
- Dynamics on $L^2(\mathbb{R}^3) \otimes L^2_{\text{sym}}(\mathbb{R}^{3N})$ with weak interaction

$$
\begin{aligned}
i\partial_t \psi_{N,t} =& H_N \psi_{N,t}\,, \\
H_N =& -\sum_{i=1}^{N} \frac{\Delta_{y_i}}{2} - \frac{\Delta_x}{2m} + \frac{1}{\rho} \sum_{1\le i<j\le N} V(y_i - y_j) \\
&+ \frac{1}{\sqrt{\rho}} \sum_{i=1}^{N} W(x - y_i)\,, \quad V, W \in C_c^\infty,\ \text{even.}
\end{aligned}
$$



Bose-Einstein condensate

Bose-Einstein condensation (BEC) if almost all bosons in same state:

$$\psi_{N,t}(y_1, \ldots, y_N) \sim \prod_{i=1}^{N} \varphi_t(y_i)\,,$$

for large ρ, $\varphi_t \in L^2(\mathbb{R}^3)$ a **one-particle state**.

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Excitations: Particle $\Theta \in \{\varphi_t\}^\perp \subset L^2(\mathbb{R}^3)$ not in the condensate.
On N-body level:

$$\underbrace{\psi_{N,t}}_{N\text{-body wave fct}} \sim \underbrace{\varphi_t \otimes_s \cdots \otimes_s \varphi_t}_{\text{Condensate particles}} \otimes_s \underbrace{\Theta_1 \otimes_s \ldots \Theta_k}_{\text{Excitations}}\,.$$

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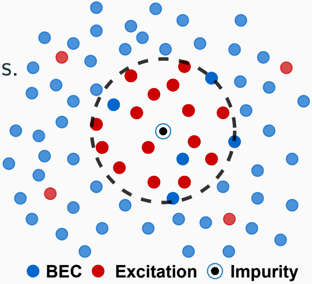
Our setting: System exhibits **Bose-Einstein condensation** with **few excitations**.

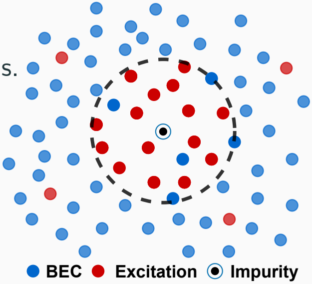
Bose Polaron: Quasi-particle of impurity and bosons. It is described by effective dynamics generated by **Bogoliubov-Fröhlich** Hamiltonian H^{BF}



$$H^{\mathsf{BF}} = H^{\mathsf{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)\,,$$

$$i\partial_t \psi_t^{\mathsf{BF}} = H^{\mathsf{BF}} \psi_t^{\mathsf{BF}}\,.$$

H^{Bog} Bogoliubov Hamiltonian, modelling free excitations. Q_t projects into $\{\varphi_t\}^{\perp}$. $a^{\#}(Q_t W_x \varphi_t)$ **creates or annihilates excitation** due to interaction of impurity with condensate.



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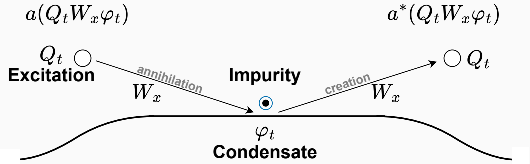
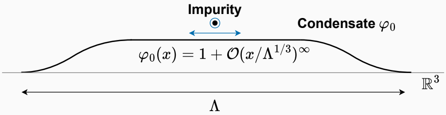


Method:

$$\underbrace{H^{\text{ex}}}_{\text{Full excitation dynamics}} = \underbrace{H^{\text{BF}}}_{\text{Polaron dynamics}} + \text{error}.$$

- Control **excitation number**
- Control **impurity localization**



Initial conditions

Result:

Conclude **effective description** of the full dynamics through H^{BF} :

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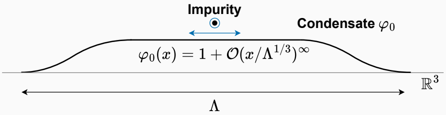

Initial conditions

Result:

Conclude **effective description** of the full dynamics through H^{BF}:
$\forall T \geq 0\ \exists C > 0$ such that $\forall \rho \geq 1$, $\Lambda = \rho^\alpha$

$$\sup_{t\in[-T,T]} \Big\| \underbrace{e^{it\sqrt{\rho}W\star 1} U_t \psi_{N,t}}_{\text{Full excitation dynamics}} - \underbrace{\psi_t^{\mathrm{BF}}}_{\text{Polaron dynamics}} \Big\| \leq C\rho^{\frac{3\alpha-1}{2}} \xrightarrow[\rho\to\infty]{\alpha<1/3} 0\,.$$

Thank you for your attention!