Triviality proof of mean-field φ_4^4-theories

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August 13th, 2024

Outline

Motivations

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- Non asymptotically free renormalizable QFTs: Quantum electrodynamics (QED), φ_4^4 theory (Higgs field with two components).
- Perturbation theory:
  - ➔ Landau pole : divergence of the running coupling constant $g(\lambda)$ at a certain energy.
  - ➔ If $\lambda_{\max} \longrightarrow +\infty$ and $g(\lambda_{\max})$ fixed: finite result if $g_{ren} = g(\lambda_{phys}) \underset{\lambda_{\max}\to+\infty}{\longrightarrow} 0$: **Theory = trivial** .
- Triviality of the φ_d^4 theory in d dimensions:
  - ➔ $d > 4$: Triviality of the continuum limit on a lattice, by Aizenman and Fröhlich in [1, 2].
  - ➔ $d = 4$: Multi-scale analysis by Aizenman and Duminil-Copin in [3].

Motivations

- Triviality of the full Standard model : unsolved important question yet $\longrightarrow$ mathematical status of the Standard model.
- Proofs of triviality : one or two-components scalar field.
- Another approach: the flow equations in the mean-field approximation in [4] by Kopper (2022) for a single component scalar field $\longrightarrow$ can also be used to analyze the $O(N)$-vector model, $N \geq 1$.

The flow equations in the mean field approximation for a $O(N)$ scalar theory

Flow equations in the mean field approximation (m.f.a.)

- Euclidean space $\mathbb{R}^4$, scalar product $\langle \cdot, \cdot \rangle$ in $L^2(\mathbb{R}^4, d^4x)$.
- $O(N)$-vector model: theory with an $O(N)$-symmetry, φ : N scalar components φ_i.
- Mean field approximation: **set all momenta to zero**.
- $f_n^N(\lambda)$: (rescaled) n-point function $\longleftrightarrow$ physical contents of the theory.
- Flow equations (FEs) in the m.f.a. : system of differential equations.
- Boundary conditions $\Longrightarrow f_n^N(\lambda)$: solutions of the FEs.
- **Inductive scheme :** From f_2^N, construct f_4^N and so on...
- Study smooth solutions f_n^N.
- In [4], Kopper studied the case $N = 1$: choice of a bare interaction lagrangian $\longrightarrow$ different types of solutions : asymptotically free solutions, trivial solution.
- In our paper [5] : improve some results of [4].

The trivial solution of the $O(N)$-model

Construction of the trivial solution

- Start with this "bare" interaction lagrangian

$$L_0(\varphi) = \int d^4x\Big(a\varphi^2(x) + b\varphi^4(x)\Big)\ , \tag{1}$$

with $\varphi^2(x) := \sum_{1\leq i\leq N} \varphi_i^2(x)$ and $\varphi^4(x) = (\varphi^2(x))^2$.

- Fixed boundary conditions in m.f.a. (at bare level):

$$f_2^N(\lambda_{\max}) = 2(2\pi)^4\lambda_{\max}^{-2}a, \quad f_4^N(\lambda_{\max}) = 4\pi^2 b, \quad f_n^N(\lambda_{\max}) = 0, \quad n \geq 6\ . \tag{2}$$

- **Triviality** : Show that $\lim_{\lambda_{\max}\to+\infty} f_4^N(\lambda_{phys}) = 0$.

- In [4] : construction of the trivial solution for small bare couplings for $N = 1$.
- **Idea**: ansatz of $f_2^1(\lambda)$ implying triviality. The ansatz is well-defined using the FEs.
- Extend the construction of a trivial solution to large bare couplings for $N = 1$ (previous proofs : no assumption on the size of the bare coupling)
- Extend it to the $O(N)$ vector model for $N > 2$ (previous proofs: restriction to one or two-components fields).

Perspectives

Future plans

- Establish a relation with perturbation theory [6, 7] (in progress)
- Beyond the mean field approximation ? Very complicated for various reasons (symmetries, momenta dependence...)

Thank you for your attention

Bibliography I

[1] Michael Aizenman. "A geometric perspective on the scaling limits of critical Ising and φ_d^4 models". In: *Current Developments in Mathematics* (2021).

[2] Jürg Fröhlich. "On the triviality of $\lambda\phi_4^4$ theories and the approach to the critical point in $d \geq 4$ dimensions". In: *Nucl.Phys B* 200 (1982), pp. 281–296. ISSN: 0550-3213.

[3] Michael Aizenman and Hugo Duminil-Copin. "Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models". In: *Annals of Mathematics* 194.1 (July 2021).

[4] Christoph Kopper. "Asymptotically Free Solutions of the Scalar Mean Field Flow Equations". In: *Annales Henri Poincaré* 23 (2022), pp. 3453–3492. ISSN: 1424-0637.

Bibliography II

[5] Christoph Kopper and Pierre Wang. “Triviality proof for mean-field φ_4^4-theories”. In: *J. Math. Phys.* (submitted).

[6] Daniel J. Amit. *Field Theory, The Renormalization Group and Critical Phenomena*. McGraw-Hill, 1978.

[7] Moshe Moshe and Jean Zinn-Justin. “Quantum field theory in the large N limit: a review”. In: *Physics Reports* 385.3-6 (Oct. 2003), pp. 69–228.